The Schur Indices of the Cuspidal Unipotent Characters of the Finite Unitary Groups

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Let $\boldsymbol{F}_q$ be a finte field with q elements of characteristic p. Let G be a connected, reductive algebraic group, defind over $\boldsymbol{F}_q$, of type $({}^2A_{n-1})$, $n \geqq 2$, and let $F: G \to G$ be the corresponding Frobenius endomorphism of G. Let G^F be the group of F-fixed points of G. According to G. Lusztig [7], G^F has a cuspidal unipotent character if and only if $n = s(s+1)/2$ for some natural number s, in which case such a character is unique. In the following, if χ is a complex irreducible character of a finite group and E is a field of characteristic 0, then $m_E(\chi)$ denotes the Schur index of χ with respect to E. The purpose of this paper is to prove the following theorem:

Theorem. Assume that $n = s(s+1)/2$ for some natural number s. Let ρ be the unique cuspidal unipotent character of G^F. Let $r = [s(s+1)/4]$ (the integral part of $s(s+1)/4$). Then, if r is even, we have $m_{\boldsymbol{Q}}(\rho) = 1$, and if r is odd, we have $m_{\boldsymbol{R}}(\rho) = m_{\boldsymbol{Q}_p}(\rho) = 2$ and $m_{\boldsymbol{Q}_l}(\rho) = 1$ for any prime number $l \neq p$.

For $n = 3, 6$, the theorem has been proved by Lusztig [6, Proposition (7.6)]. Our proof below is on the same lines as his method.

The unipotent characters of G^F can be naturally parametrized by the irreducible characters of the symmetric group S_n (see below) and the latter can be parametrized by the partitions of n (see, e.g. [5]). Using the deformation theory of Howlett and Lehrer [3, cf. Th. (5.9)] and a result of Benson and Curtis [1] as well as that of Lusztig [8, pp. 33–5] and reasoning like on [**10**, p. 297], we get (cf. [7]):

Corollary. Let $\rho = \rho_\alpha$ be the unipotent character of G^F corresponding to a partition α of n. Let $n' =$ the number of squares in the Young diagram of α which have an odd hook length minus the number of squares which have an even hook length. Then we have $m_{\boldsymbol{Q}}(\rho) = 1$ if $[n'/2]$ is even. If $[n'/2]$ is odd, then we have $m_{\boldsymbol{R}}(\rho) = m_{\boldsymbol{Q}_p}(\rho) = 2$ and $m_{\boldsymbol{Q}_l}(\rho) = 1$ for any prime number $l \neq p$.

Using this corollary, we can determine the rationality of the generalized Gelfand-Graev representations of $U_n(\boldsymbol{F}_q)$ (Kawanaka [4]) when $p = 2$ (see [11]).

Proof of the theorem. Let G and F be as above. Let B^* and T^* be respectively an F-stable Borel subgroup of G and an F-stable maximal torus of B^*. Let $W = N_G(T^*)/T^*$ be the Weyl group of G with respect to T^*. Let $l(\ )$ be the length function on W with respect to the simple reflections in W determined by B^*. For an element w of W, let $X(w)$ be the variety of all Borel subgroups B of G such that B and FB are in relative position w (see [2]). G^F acts on $X(w)$ by conjugation. Let l be a fixed prime number $\neq p$, and let $\bar{\boldsymbol{Q}}_l$ be an algebraic closure of $\boldsymbol{Q}_l$. Then, for $w \in W$, we define a virtual module R_w of G^F over $\bar{\boldsymbol{Q}}_l$ by

$$R_w = \sum_{i=0}^{2l(w)} (-1)^i H_c^i(X(w), \bar{\boldsymbol{Q}}_l),$$

which we regard as a generalized character of G^F with values in $\bar{\boldsymbol{Q}}_l$ (Deligne-Lusztig [2]). Let w_0 be the longest element of $W = S_n$. For each irreducible character χ of W, set

$$\rho_\chi = \frac{1}{|W|} \sum_{w \in W} \chi(ww_0) R_w.$$

Then $\pm \rho_\chi$ are precisely the mutually different unipotent characters of G^F (Lusztig-Srinivasan [9]). If $n = s(s+1)/2$ and if χ is the irreducible character of S_n corresponding to the partition $(s, s-1, s-2, \dots, 3, 2, 1)$ of n, then $\pm \rho_\chi$ is the cuspidal unipotent character of G^F ([7]).

Let χ be an irreducible character of S_n. Then there is an element $w \in W$ such that $X(w) = \pm 1$. [In fact, let Γ be the regular graph of the partition of n corresponding to χ. For $i \geqq 1$, let n_i be the number defined as (the number of nodes in the i-th row of Γ) + (the number of nodes in the i-th colum of Γ) $- 2i + 1$. Then, by Theorem II of [5], we see that $\chi(w) = \pm 1$, where w is an element of W contained in the class of W corresponding to the partition $(n_1, n_2, n_3, \dots)$ of n.]

Let $\rho = \pm \rho_\chi$ be the unipotent character of G^F corresponding to an irreducible character χ of S_n. Then we have $(R_w, \rho)_{G^F} = \pm \chi(ww_0) = \pm 1$ for some element $w \in W$. We fix such an element w. The G^F-action on $H_c^i(X(w), \bar{\boldsymbol{Q}}_l) = H_c^i(X(w), \boldsymbol{Q}_l) \otimes_{\boldsymbol{Q}_l} \bar{\boldsymbol{Q}}_l$ is induced by the G^F-action on $H_c^i(X(w), \boldsymbol{Q}_l)$. Therefore each $\bar{\boldsymbol{Q}}_l[G^F]$-module $H_c^i = H_c^i(X(w), \bar{\boldsymbol{Q}}_l)$ is defined over $\boldsymbol{Q}_l$. Therefore, by the property of the Schur index, we have $m_{\boldsymbol{Q}_l}(\rho) \mid (H_c^i, \rho)_{G^F}$ for each i. Therefore we have $m_{\boldsymbol{Q}_l}(\rho) = 1$.

As each R_w is integral-valued, ρ is $\boldsymbol{Q}$-valued. Therefore, as l is any prime number $\neq p$, by Hasses's sum formula, we must have $m_{\boldsymbol{R}}(\rho) = m_{\boldsymbol{Q}_p}(\rho)$. Let us determine $m_{\boldsymbol{R}}(\rho)$. As ρ is $\boldsymbol{Q}$-valued, by a theorem of Frobenius-Schur (see Serre [12]), there is a simple $\mathbf{C}[G^F]$-module V_0 which affords ρ, with G^F-equivariant non-degenerate bilinear form f_0 with values in $\mathbf{C}$; we have $m_{\boldsymbol{R}}(\rho) = 1$ (resp. 2) if f_0 is symmetric (resp. anti-symmetric). Let us find such a module V_0. In the following, we shall follow the argument of Lusztig in [8, pp. 25–26].

For any sequence $\underline{s} = (s_1, s_2, \dots, s_k)$ of simple reflections in W, let $X(\underline{s})$ be the variety of all sequences $(B_0, B_1, \dots, B_k)$ of Borel subgroups of G such that B_{i-1} and B_i are in relative position s_i for $1 \leqq i \leqq k$ and $FB_k = B_0$. Then G^F acts on $X(\underline{s})$ by conjugation on each factor, hence we can consider a virtual module $R_{\underline{s}} = \sum_i (-1)^i H_c^i(X(\underline{s}), \bar{\boldsymbol{Q}}_l)$. Let $\underline{s}$ be a sequence with minimum possible k such that $(R_{\underline{s}}, \rho)_{G^F}$ is odd. Then, by the argument similar to that in [8, p. 25 line 26 – p. 26 line 4], we see that $l(w') = k$ where $w' = s_1 s_2 \cdots s_k$. Thus the correspondence $(B_0, B_1, \dots, B_k) \to (B_0, B_k)$ defines an isomorphism of $X(\underline{s})$ with $X(w')$ (by Bruhat decomposition). Let $\bar{X}$ be the projective variety consisting of all sequences $(B_0, B_1, \dots, B_k)$ of Borel subgroups of G such that, for $1 \leqq i \leqq k$, B_{i-1} and B_i are in relative position s_i or e (the unit of W) and that $FB_k = B_0$. Then $\bar{X}$ is smooth of pure dimension k, $X(w') = X(\underline{s})$ is an open subvariety of $\bar{X}$, and the complement $\bar{X} - X(\underline{s})$ is the disjoint union of locally closed subvarieties $X(\tilde{\underline{s}})$, where $\tilde{\underline{s}}$ runs over certain subsequences of $\underline{s}$ other than $\underline{s}$ (Deligne-Lusztig [2]). The inclusions $X(w') \hookrightarrow \bar{X} \hookleftarrow \bar{X} - X(w')$ give rise to a long exact sequence of cohomologies, and using the ones arising from the locally closed disjoint union $\bar{X} - X(w') = \coprod X(\tilde{\underline{s}})$, we get

$$\bar{R} = \sum_{i=0}^{2k} (-1)^i H^i(\bar{X}, \bar{\boldsymbol{Q}}_l) = R_{w'} + \sum_{\tilde{\underline{s}}} R_{\tilde{\underline{s}}},$$

where in the right hand side of the second equality, the sum is taken over certain subsequences $\tilde{\underline{s}}$ of $\underline{s}$ other than $\underline{s}$. (As before l is any fixed prime number $\neq p$.) By the assumption on $\underline{s}$, $(R_{\tilde{\underline{s}}}, \rho)_{G^F}$ is even for each such subsequence $\tilde{\underline{s}}$ of $\underline{s}$, hence $(\bar{R}, \rho)_{G^F}$ must be odd. As ρ is selfdual, by the Poincaré duality of etale cohomology, we see that, for each $i > 0$, $(H^i(\bar{X}, \bar{\boldsymbol{Q}}_l), \rho)_{G^F} = (H^{2k-i}(\bar{X}, \bar{\boldsymbol{Q}}_l), \rho)_{G^F}$. Therefore, we conclude that $(H^k(\bar{X}, \bar{\boldsymbol{Q}}_l), \rho)_{G^F}$ is odd. Let V be the ρ-isotropic part of $H^k(\bar{X}, \bar{\boldsymbol{Q}}_l)$. Then the Poincaré duality on $H^k(\bar{X}, \bar{\boldsymbol{Q}}_l)$ induces on V a nondegenerate bilinear mapping f with values in $\bar{\boldsymbol{Q}}_l$. f is compatible with the action of G^F. f is symmetric (resp. anti-symmetric) if k is even (resp. odd).

Let us show that there is a simple submodule V_0 of V such that The restriction of f to V_0 is nondegenerate. In fact, suppose, on the contrary, that no such submodules exist. Then, for any simple submodule V' of V, we must have $V' \subset V'^{\perp}$, where $V'^{\perp} = \{y \in V \mid f(x, y) = 0 \text{ for all } x \in V'\}$. [We note that, as ρ is selfdual, $\mathrm{Hom}(V', \bar{\boldsymbol{Q}}_l)$ is isomorphic to V' as G^F-module.] For any submodule V' of V, let m' be the multiplicity of ρ in V'. Let V' be a submodule of V with minimum possible odd m' such that the restriction f' of f to V' is nondegenerate. Let V_1 be a simple submodule of V'. As f' is nondegenerate, we have $\dim_{\bar{\boldsymbol{Q}}_l} V_1^{\perp} + \dim_{\bar{\boldsymbol{Q}}_l} V_1 = \dim_{\bar{\boldsymbol{Q}}_l} V'$, where $V_1^{\perp}$ is the subspace of V' which is orthogonal to V_1 with respect to f'. $V_1^{\perp}$ is G^F-stable. As V' is a semisimple $\bar{\boldsymbol{Q}}_l[G^F]$-module, there is a submodule V_2 of V' such that $V' = V_1^{\perp} \oplus V_2$. V_2 is simple since $\dim_{\bar{\boldsymbol{Q}}_l} V_2 = \dim_{\bar{\boldsymbol{Q}}_l} V_1$. We note that $V_1 \subset V_1^{\perp}$. We see that $V_1^{\perp} = V_1 \oplus M$, where $M = V_1^{\perp} \cap V_2^{\perp} = (V_1 \oplus V_2)^{\perp}$, and this is an orthogonal decomposition of $V_1^{\perp}$. It is easy to see that the restriction of f' to M is nondegenerate. But, as $(M, \rho)_{G^F} = m' - 2$ is odd, this contradicts to the minimality of m'.

Let V_0 be a simple submodule of V such that the restriction f_0 of f to V_0 is nondegenerate. As $\bar{\boldsymbol{Q}}_l$ is isomorphic to $\boldsymbol{C}$, we may regard f_0 as a form with values in $\boldsymbol{C}$. Therefore, by the theorem of Frobenius-Schur, we have $m_{\boldsymbol{R}}(\rho) = 1$ (resp. $m_{\boldsymbol{R}}(\rho) = 2$) if k is even (resp. odd). Suppose that $n = s(s+1)/2$, and that ρ is cuspidal. Then $r = [s(s+1)/4] = [n/2]$ is equal to the semisimple $\boldsymbol{F}_q$-rank of G. As ρ is cuspidal and $(R_{w'}, \rho)_{G^F} \neq 0$, we must have $(-1)^k = (-1)^r$. This completes the proof of the theorem.

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