Fermat Varieties of Hodge-Witt Type

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(Communicated by Shokichi IYANAGA, M. J. A., March 13, 1995)

§1. Introduction. Let X be a smooth projective variety over a perfect field k of characteristic $p > 0$, and W be the ring of Witt vectors on k. If X is of Hodge-Witt in all degrees in the sense of (4.6) in Chapitre IV of Illusie-Raynaud [4], that is, if Hodge-Witt cohomology groups $H^j(X, W\Omega_X^i)$ are of finite type over W for all (i, j), then we say that X is *of Hodge-Witt type.* E.g. if X is a curve, X is of Hodge-Witt type (cf. Serre [7], Chapter II of Illusie [2]). When X is a smooth complete intersection in a projective space, we know, through Suwa [9], that X is of Hodge-Witt type if the "niveau de Hodge" of X in the sense of Deligne [1] or Rapoport [6] is at most one. If $V_n(m)$ means a hypersurface of degree m in an $(n+1)$-dimensional projective space $\boldsymbol{P}^{n+1}$, each $V_n(m)$ for "$n > 0$, $m = 2$", "$n = 2$, $m = 3$", "$n = 3$, $m = 3$", "$n = 3$, $m = 4$", "$n = 5$, $m = 3$" is of the "niveau de Hodge" ≤ 1 according to Rapoport [6], 2, Table 1. Moreover we know that if X is *ordinary* in the sense of (4.12) in Chapitre IV of Illusie-Raynaud [4] then X is of Hodge-Witt type.

We are concerned with the smooth hypersurface S of degree $m > 0$ defined by an equation:

$$a_0x_0^m + a_1x_1^m + \cdots + a_{n+1}x_{n+1}^m = 0$$

(the a_i are in k, and not 0) over a finite field k of characteristic $p > 0 (p \nmid m)$ in $\boldsymbol{P}^{n+1}$ of which homogeneous coordinates are $x_0, x_1, \ldots, x_{n+1}$. Then, over an algebraic closure of k, the hypersurface S is isomorphic to the Fermat variety $F_{n,m,p}$ of dimension $n > 0$, degree $m > 0$ defined by

$$x_0^m + x_1^m + \cdots + x_{n+1}^m = 0$$

in $\boldsymbol{P}^{n+1}$

To show that S is of Hodge-Witt type, it is sufficient to show that $F_{n,m,p}$ is of this type in degree n by Suwa [9].

Now we consider the Fermat variety $F_{n,m,p}$ with $\{n, m, p\} (n > 0, m > 0, p \nmid m)$. From what we have said above, we know the followings:

Case (1) $\{n, m, p\} = \{n, 1, p\}$ or $\{n, 2, p\}$;

$F_{n,1,p}$ *and* $F_{n,2,p}$ *are ordinary and hence of Hodge-Witt type.*

Case (2) $\{n, m, p\} = \{1, m, p\}$;

$F_{1,m,p}$ *is of Hodge-Witt type.*

Case (3) $\{n, m, p\}$ *with* $p \equiv 1 \bmod m$;

$F_{n,m,p}$ *is ordinary and hence of Hodge-Witt type.*

Case (4)

$$\{n, m, p\} = \begin{cases} \{2,3, p\} & \text{with } p \equiv 2 \bmod 3, \\ \{3,3, p\} & \text{with } p \equiv 2 \bmod 3, \\ \{3,4, p\} & \text{with } p \equiv 3 \bmod 4, \\ \{5,3, p\} & \text{with } p \equiv 2 \bmod 3; \end{cases}$$

$F_{2,3,p}$, $F_{3,3,p}$, $F_{3,4,p}$ *and* $F_{5,3,p}$ *are of Hodge-Witt type.*

In addition we have obtained the following result through Suwa's criterion (see §2).

Theorem. *Let the triplet* $\{n, m, p\}$ *of integers* $n > 1$, $m > 2$, *and a prime number* p *with* $p \nmid m$ *and* $p \not\equiv 1 \pmod m$ *be given. Then we have the following assertion:*

$F_{n,m,p}$ *is of Hodge-Witt type*

if and only if

$\{n, m, p\}$ *is in the above case* (4) *or in the case* (5) "$n = 2$, $m = 7$ *and* $p \equiv 2,4 \pmod 7$".

The assertion for $n = 2$, $m \geq 4$ in the Theorem has been conjectured by N. Suwa.

The author wishes to express his hearty thanks to Prof. N. Suwa, who has communicated to him this conjecture and the known results recalled at the beginning of this paper.

§2. The set $\mathcal{W}$. Let p be a prime number. And let the triplet $\{n, m, p\}$ of integers with $n > 0$, $m > 0$, $p \nmid m$ be given. For $w = (w_0, w_1, \ldots, w_{n+1}) \in \boldsymbol{Z}^{n+2}$, let the integer $|w|$ be defined by

$$|w| = \sum_{j=0}^{n+1} w_j.$$

Moreover, we set

$$\mathcal{W} = \{w \in \boldsymbol{Z}^{n+2}; 0 < w_j < m (j = 0,1,2,\ldots, n+1), |w| \equiv 0 \bmod m\},$$

$$\mathcal{W}_i = \{w \in \mathcal{W}; |w| = (i+1)m\} (i = 0,1,2,\ldots).$$

Then we have

$$\mathscr{W} = \cup_{i=0}^{n} \mathscr{W}_i.$$

Let γ be an integer relatively prime to m. We consider the action $\gamma\cdot$ on $\mathscr{W}$, defined by

$$\gamma\cdot w = (\{\gamma w_0\}_m, \{\gamma w_1\}_m, \dots, \{\gamma w_{n+1}\}_m)$$

for $w \in \mathscr{W}$, where each $\{\gamma w_j\}_m$ denotes the remainder of dividing γw_j by m. Then, for two integers γ, γ' relatively prime to m, we have $\gamma\cdot = \gamma'\cdot$ if $\gamma \equiv \gamma' \bmod m$, $(\gamma\gamma')\cdot = \gamma\cdot(\gamma'\cdot)$ on $\mathscr{W}$.

Let the integer $\| w \|$ be defined by $\| w \| = \frac{| w |}{m} - 1$ for $w \in \mathscr{W}$ in n and m. Then we obtain

$$w \in \mathscr{W}_i \Leftrightarrow \| w \| = i.$$

The following is a key lemma for our proof of the Theorem.

Lemma (Suwa's criterion). *Let* $\{n, m, p\}$ *with* $n > 1, m > 2, p \not| m$ *and* $p \not\equiv 1 \pmod m$ *be given. Let* $\mathscr{W}$ *be the set defined as above.*

Then the following conditions are equivalent:

(*i*) *The Fermat variety* $F_{n,m,p}$ *is of Hodge-Witt type.*

(*ii*) *The triplet* $\{n, m, p\}$ *satisfies the assertion*

"At each element w *in* $\mathscr{W}$, *we have*

$\| p^{\alpha}\cdot w \| = \| w \|$ *for all non-negative integers* α

or

$\| p^{\alpha}\cdot w \| - \| w \| \in \{0,1\}$ *for all non-negative integers* α

and $\| p^{\alpha}\cdot w \| = \| w \| + 1$ *for some positive integer* α

or

$\| p^{\alpha}\cdot w \| - \| w \| \in \{0, -1\}$ *for all non-negative integers* α

and $\| p^{\alpha}\cdot w \| = \| w \| - 1$ *for some positive integer* α".

This lemma follows from the theory of Hodge-Witt cohomology applied to the case of the Fermat varieties (cf. Illusie [3], Koblitz [5], Shioda-Katsura [8]).

§3. A sketch of proof. Through the combinatorial arithmetic on $\mathscr{W}$ for $\{n, m, p\}$, we have been able to give a proof of the Theorem. As to the "if" part, it is directly seen that "the condition (ii) in Lemma" (abridged L(ii)) holds in cases (4) and (5) of $\{n, m, p\}$. As to the "only if" part, which means that L(ii) holds only in cases (4) and (5), we can show that its proof is reduced to the fact that L(ii) does not hold in the following cases A,B.

Case A: $\{n, m, p\} = \{2,4, p\}\,(p \equiv 3 \bmod 4)$, $\{2,5, p\}\ (p \not\equiv 1 \bmod 5)$, $\{2,6, p\}\,(p \equiv 5 \bmod 6)$, $\{2,7, p\}\,(p \equiv 3,5,6 \bmod 7)$, $\{2,8,p\}\,(p \not\equiv 1 \bmod 8)$, $\{3,5, p\}\,(p \not\equiv 1 \bmod 5)$, $\{3,6, p\}\,(p \equiv 5 \bmod 6)$, $\{3,7, p)\ (p \not\equiv 1 \bmod 7)$, $\{3,8, p\}\,(p \not\equiv 1 \bmod 8)$, $\{4,3, p\}\,(p \equiv 2 \bmod 3)$, $\{4,7, p\}\,(p \equiv 2,4 \bmod 7)$, $\{5,4, p\}\,(p \equiv 3 \bmod 4)$, $\{7,3, p\}\,(p \equiv 2 \bmod 3)$.

Case B: $\{n, m, p\} = \{2, m, p\}$, $\{3, m, p\}$, where $m \geq 9$ and $p \not\equiv 1 \bmod m$.

In case A, it is not difficult to verify that L(ii) does not hold in each of enumerated cases. In case B, the proof of the non-validity of L(ii) is easy if "$n = 2$ *and* $\{p\}_m = \frac{m+1}{2}$ (m is odd)", or if "$n = 2,3$ *and* $\{p\}_m = 2$". Otherwise, we use the following lemma.

Lemma B. *For* $9 \leq m \in \boldsymbol{Z}$, *we have*

(*a*): $\left[3, \frac{m}{2}\right) \subset \cup_{2 \leq k \leq [(m-1)/3], k \in \boldsymbol{Z}} \left[\frac{2m}{3k}, \frac{m}{k}\right)$,

(*b*): $\left[\frac{m}{2} + 1, m\right) \subset \cup_{1 \leq k \leq [(m-1)/3], k \in \boldsymbol{Z}} \left[\frac{(3k+1)m}{6k}, \frac{(k+1)m}{2k}\right)$,

(*c*): $[3, m) \subset \cup_{1 \leq k \leq [(m-1)/4], k \in \boldsymbol{Z}} \left[\frac{m}{2k}, \frac{m}{k}\right)$.

We consider the following subcases of Case B:

"(*a*) $n = 2$ *and* $3 \leq \{p\}_m < \frac{m}{2}$",

"(*b*) $n = 2$ *and* $\frac{m}{2} + 1 \leq \{p\}_m < m$",

"(*c*) $n = 3$ *and* $3 \leq \{p\}_m < m$".

In applying (*a*), (*b*), (*c*) of Lemma B respectively, we see that L(ii) does not hold.

The detailed account will be published elsewhere.

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