Accessibility of Infinite Dimensional Brownian Motion to Holomorphically Exceptional Set*)

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1. Introduction. In [6], we introduced the notion of *holomorphically exceptional sets* of the complex Wiener space. In particular, we pointed out the following remarkable relation between holomorphically exceptional sets and the standard Brownian motion $(Z_t)_{t\geq 0}$ on the complex Wiener space: *Z_t does not hit a holomorphically exceptional set until time* 1 *almost surely.*

In any finite dimensional space, if the Brownian motion does not hit a certain set until time 1 almost surely, neither does it after time 1. So one may guess that the infinite dimensional Brownian motion never hits a holomorphically exceptional set after time 1, either.

But we will show in the present paper that the above guess is false. That is, we will construct a holomorphically exceptional set which the Brownian motion $(Z_t)_{t\geq 0}$ hits after a certain time $t_0 > 1$ almost surely.

The reason why such an example can exist lies essentially in a fact that the distributions of $(Z_t)_{t\geq 0}$ at different times are mutually singular.

2. Presentation of Theorem. Let (B, H, μ) be a *real* abstract Wiener space, i.e., B is a real separable Banach space (whose dimension is infinite), H is a real separable Hilbert space continuously and densely imbedded in B and μ is a Gaussian measure satisfying

$$\int_B \exp(\sqrt{-1}\langle z, l\rangle)\mu(dz) = \exp\left(-\frac{1}{4}\| l \|^2_{H^*}\right) \qquad l \in B^* \subset H^*.$$

We introduce an *almost complex structure* $J : B \to B$ which is an isometry such that $J^2 = -1$ and that the restriction $J|_H : H \to H$ is also an isometry. The abstract Wiener space (B, H, μ) endowed with the almost complex stucture J is called an *almost complex abstract Wiener space* and denoted by (B, H, μ, J).

Let B^{*C} be the complexification of the dual space B^*. Then define

$$B^{*(1,0)} := \{\varphi \in B^{*C} \mid J^*\varphi = \sqrt{-1}\varphi\},$$
$$B^{*(0,1)} := \{\varphi \in B^{*C} \mid J^*\varphi = -\sqrt{-1}\varphi\}.$$

In other words, $B^{*(1,0)}$ is the space of bounded *complex linear* functionals on B and $B^{*(0,1)}$ is the space of bounded *complex anti-linear* functionals on B. We see that $B^{*C} = B^{*(1,0)} \oplus B^{*(0,1)}$. The Hilbert spaces H^{*C}, $H^{*(1,0)}$ and $H^{*(0,1)}$ are similarly defined.

Definition. 1. A function $G : B \to \boldsymbol{C}$ is called a *holomorphic polynomial,* if it is expressed in the form

(1) $G(z) = g(\langle z, \varphi_1\rangle, \ldots, \langle z, \varphi_n\rangle),\ z \in B,$

where $n \in \boldsymbol{N}$, $g : \boldsymbol{C}^n \to \boldsymbol{C}$ is a polynomial with complex coefficients and $\varphi_1, \ldots, \varphi_n \in B^{*(1,0)}$ The class of all holomorphic polynomials is denoted by $\mathscr{P}_h$.

Definition. 2. Let $p \in (1, \infty)$. For a sequence $\{G_n\} \subset \mathscr{P}_h$ such that $\sum_n \| G_n \|_{L^p(\mu)} < \infty$, we define a subset $N^p(\{G_n\})$ of B by

(2) $N^p(\{G_n\}) := \{z \in B \mid \sum_n | G_n(z) | = \infty\}.$

A set $A \subset B$ is called an *L^p-holomorphically exceptional set,* if it is a subset of a set of the type $N^p(\{G_n\})$. We denote the class of all L^p-holomorphically exceptional sets by $\mathscr{N}^p_h$. If an assertion holds outside of an L^p-holomorphically exceptional set, we say that it holds "$a.e.(\mathscr{N}^p_h)$".

Let $(Z_t)_{t\geq 0}$ be a B-valued independent increment process defined on a probability space $(\Omega, \mathscr{F}, P)$ such that $Z_0 = 0$ and the distribution of $Z_t - Z_s$, $t > s$, is μ_{t-s}, where $\mu_r(\cdot) := \mu(\cdot / \sqrt{r})$. Then the process $(Z_t)_{t\geq 0}$ becomes a diffusion process on B and it is called a *B-valued Brownian motion* (see, for example, [3]).

In [6], it is known that $(Z_t)_{t\geq 0}$ does not hit any L^p-holomorphically exceptional set until time 1 almost surely.

Theorem. *There exists an L^2-holomorphically exceptional set $A \subset B$ such that*

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$$1 < \sigma_A := \inf\{t \geq 0 \mid Z_t \in A\} < \infty \quad a.s.$$

We construct the set A as follows: Let $\{\varphi_n\}_{n=1}^{\infty} \subset B^{*(1,0)}$ be an orthonormal system of $H^{*(1,0)}$. Then put

$$(3) \qquad G_n(z) := \frac{1}{n^2} \prod_{j=\frac{n(n-1)}{2}+1}^{\frac{n(n+1)}{2}} \langle z, \varphi_j \rangle, \; n = 1,2,\ldots$$

Note that $\{G_n\}_{n=1}^{\infty}$ is a sequence of independent random variables under each probability measure μ_t, $t > 0$ and that $\| G_n \|_{L^2(\mu)} = 1/n^2$. Finally we define A by

$$(4) \qquad A := N^2(\{G_n\}).$$

Then we will prove that

$$\sigma_A = e^{\gamma}, \quad \text{a.s.},$$

where

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.57721\ldots$$

is Euler's constant.

Remark. If $t \neq t'$ then μ_t and $\mu_{t'}$ are mutually singular, and hence there exists a set K such that

$$\begin{cases} \mu_t(K) = 0, & \text{if } 0 \leq t \leq 1, \\ \mu_t(K) = 1, & \text{if } 1 < t. \end{cases}$$

But, we do not know in general whether $1 \leq \sigma_K$ or not for such K.

3. Proof of Theorem. In this section, we always assume that A is the L^2-holomorphically exceptional set of B defined by (4).

Lemma 1. *For each $t \geq e^{\gamma}$, we have $\mu_t(A) = 1$*

Lemma 1 means that $Z_t \in A$, a.s., if $t \geq e^{\gamma}$, and hence $\sigma_A \leq e^{\gamma}$, a.s. This lemma immediately follows from the following lemma.

Lemma 2. *Let $\xi_1, \xi_2, \ldots$ be a sequence of $[0, \infty)$-valued i.i.d. random variables with distribution $2r\exp(-r^2)dr$. Put*

$$g_n := \frac{1}{n^2} \prod_{j=\frac{n(n-1)}{2}+1}^{\frac{n(n+1)}{2}} \xi_j, \; n = 1,2,\ldots$$

Then if $t \geq e^{\gamma}$, we have

$$\sum_{n=1}^{\infty} t^{n/2} g_n = \infty, \quad a.s.$$

Proof. In fact, we have

$$(5) \qquad \varlimsup_{n\to\infty} e^{\gamma n/2} g_n = \infty, \quad \text{a.s.},$$

which we will show below.

We first rewrite $\log g_n$ as

$$\log g_n = -\frac{\gamma n}{2} + S_n - 2\log n,$$

where

$$S_n := \sum_{j=\frac{n(n-1)}{2}+1}^{\frac{n(n+1)}{2}} \Xi_j, \quad \Xi_j := \log \xi_j + \frac{\gamma}{2}.$$

Note that $\{\Xi_j\}_{j=1}^{\infty}$ is a sequence of i.i.d. random variables with mean 0 and variance $v := \mathrm{Var}(\Xi_1)$ $\left(= \frac{1}{4}(\Gamma''(1) - \gamma^2) > 0\right)$, which are indeed computed by using the equality $\gamma = -\Gamma'(1)$ (see, for example, [2]). Then we have

$$(6) \qquad e^{\gamma n/2} g_n = n^{S_n/\log n - 2}.$$

According to the central limit theorem, we see

$$\lim_{n\to\infty} P\left(\frac{S_n}{\sqrt{n}} \geq 1\right) = \lim_{n\to\infty} P\left(\frac{\sum_{j=1}^{n} \Xi_j}{\sqrt{n}} \geq 1\right) = \int_1^{\infty} \frac{1}{\sqrt{2\pi v}} e^{-x^2/2v} dx > 0,$$

and hence

$$\sum_{n=1}^{\infty} P\left(\frac{S_n}{\sqrt{n}} \geq 1\right) = \infty.$$

Since $\{\{S_n/\sqrt{n} \geq 1\}\}_{n=1}^{\infty}$ are independent events, the second Borel-Cantelli lemma implies that

$$P\left(\frac{S_n}{\sqrt{n}} \geq 1, \text{ infinitely often}\right) = 1.$$

Thus we see

$$P\left(\varlimsup_{n\to\infty} \frac{S_n}{\log n} = \infty\right) = 1,$$

and hence by (6) we finally have (5).

Now that we have seen $\sigma_A \leq e^{\gamma}$, we will prove the opposite inequality:

Lemma 3. *$\sigma_A \geq e^{\gamma}$, a.s.*

Since A is a holomorphically exceptional set, it is known that $\sigma_A \geq 1$ by [6]. To get more precise estimate as in Lemma 3, we need the following lemma.

Lemma 4. *Let $0 < T < e^{\gamma}$. Then there exists $0 < p < 1$ such that*

$$T^{p/2} \Gamma\left(\frac{p}{2} + 1\right) < 1.$$

Proof. We first show the following inequality:

$$(7) \qquad \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-x/n} \geq \exp\left(-\frac{\pi^2 x^2}{6}\right) \quad 0 < x \leq 1.$$

To do this, we note two simple facts:

$$(8) \quad (1 + x)e^{-x} = 1 - x^2 \int_0^1 s e^{-xs} ds, \quad x \in \boldsymbol{R}$$

$$(9) \qquad \prod_{n=1}^{\infty} (1 - a_n) \geq \exp\left(-\sum_{n=1}^{\infty} \frac{a_n}{1 - a_n}\right), \quad 0 \leq a_n < 1, \; n = 1,2,\ldots$$

These two facts imply for any $0 < x \leq 1$ that

$$\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n}=\prod_{n=1}^{\infty}\left(1-x_n^2\int_0^1 se^{-x_n s}ds\right)$$

where $x_n := x/n$

$$\geq \exp\left(-\sum_{n=1}^{\infty}\frac{x_n^2\int_0^1 se^{-x_n s}ds}{1-x_n^2\int_0^1 se^{-x_n s}ds}\right)$$

$$\geq \exp\left(-\sum_{n=1}^{\infty}x_n^2\right)$$

$$=\exp\left(-x^2\sum_{n=1}^{\infty}\frac{1}{n^2}\right)=\exp\left(-\frac{\pi^2x^2}{6}\right),$$

thus we obtain (7).

Since we have assumed $0<T<e^{\gamma}$, and hence $\gamma-\log T>0$, we can take $0<p<1$ such that

$$\gamma-\log T-\frac{\pi^2}{6}\frac{p}{2}>0. \tag{10}$$

Then we see that

$$T^{p/2}\Gamma\left(\frac{p}{2}+1\right)$$
$$=e^{(p/2)\log T}\Gamma\left(\frac{p}{2}+1\right)$$
$$=e^{-(p/2)(\gamma-\log T)}e^{\gamma p/2}\Gamma\left(\frac{p}{2}+1\right).$$

Noting (7), (10) and Weierstrass's formula

$$\frac{1}{\Gamma(x+1)}=e^{\gamma x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n},\quad x>0,$$

we see

$$T^{p/2}\Gamma\left(\frac{p}{2}+1\right)$$
$$=e^{-(p/2)(\gamma-\log T)}\frac{1}{\prod_{n=1}^{\infty}\left(1+\frac{p}{2n}\right)e^{-p/2n}}$$
$$\leq e^{-(p/2)(\gamma-\log T)}\exp\left(\frac{\pi^2}{6}\left(\frac{p}{2}\right)^2\right)$$
$$=\exp\left(-\frac{p}{2}\left(\gamma-\log T-\frac{\pi^2}{6}\frac{p}{2}\right)\right)<1.$$

Thus the proof of the lemma is complete.

Proof of Lemma 3. Let $0<p<1$ be as in Lemma 4. By Minkowski's inequality, we have

$$\left(\sum_{n=1}^{\infty}|G_n(Z_t)|\right)^p\leq\sum_{n=1}^{\infty}|G_n(Z_t)|^p$$
$$\leq\sum_{n=1}^{\infty}\sup_{0\leq s\leq T}|G_n(Z_s)|^p$$
$$0\leq t\leq T.$$

Therefore,

$$\sup_{0\leq t\leq T}\left(\sum_{n=1}^{\infty}|G_n(Z_t)|\right)^p\leq\sum_{n=1}^{\infty}\sup_{0\leq t\leq T}|G_n(Z_t)|^p,$$

and hence,

$$E\left[\sup_{0\leq t\leq T}\left(\sum_{n=1}^{\infty}|G_n(Z_t)|\right)^p\right]\leq\sum_{n=1}^{\infty}E\left[\sup_{0\leq t\leq T}|G_n(Z_t)|^p\right]. \tag{11}$$

Now, let p' be such that $0<p'<p$. Since $(G_n(Z_t))_{t\geq0}$ is a conformal martingale, $(|G_n(Z_t)|^{p'})_{t\geq0}$ is a submartingale (see [1]). By Doob's inequality, we then have

$$E\left[\sup_{0\leq t\leq T}|G_n(Z_t)|^p\right]$$
$$=E\left[\left(\sup_{0\leq t\leq T}|G_n(Z_t)|^{p'}\right)^{p/p'}\right]$$
$$\leq\left(\frac{p}{p-p'}\right)^{p/p'}E[|G_n(Z_T)|^p].$$

Combining this with (11), we have

$$E\left[\sup_{0\leq t\leq T}\left(\sum_{n=1}^{\infty}|G_n(Z_t)|\right)^p\right]\leq\left(\frac{p}{p-p'}\right)^{p/p'}\sum_{n=1}^{\infty}E[|G_n(Z_T)|^p]. \tag{12}$$

By the definition (3) of $G_n(z)$ and $P(Z_T\in\cdot)=\mu(\sqrt{T}z\in\cdot)$, we see

$$E[|G_n(Z_T)|^p]$$
$$=\left(\frac{1}{n^2}\right)^p\int_B T^{np/2}\prod_{j=n(n-1)/2+1}^{n(n+1)/2}|\langle z,\varphi_j\rangle|^p\mu(dz)$$
$$=\left(\frac{1}{n^2}\right)^p T^{np/2}\left(\int_B|\langle z,\varphi_1\rangle|^p\mu(dz)\right)^n.$$

Here, the last integral is calculated as

$$\int_B|\langle z,\varphi_1\rangle|^p\mu(dz)$$
$$=\iint_{\boldsymbol{R}^2}(x^2+y^2)^{p/2}\frac{1}{\pi}e^{-(x^2+y^2)}dxdy$$
$$=\int_0^{\infty}\int_0^{2\pi}r^p\frac{1}{\pi}e^{-r^2}rdrd\theta$$
$$=\int_0^{\infty}2r^pe^{-r^2}rdr$$
$$=\int_0^{\infty}s^{p/2}e^{-s}ds=\Gamma\left(\frac{p}{2}+1\right).$$

So we have

$$E[|G_n(Z_T)|^p]=\left(\frac{1}{n^2}\right)^pT^{np/2}\Gamma\left(\frac{p}{2}+1\right)^n$$
$$=\frac{1}{n^{2p}}\left(T^{p/2}\Gamma\left(\frac{p}{2}+1\right)\right)^n.$$

Then by virtue of (11), we see

$$E\left[\sup_{0\leq t\leq T}\left(\sum_{n=1}^{\infty}|G_n(Z_t)|\right)^p\right]$$
$$\leq\left(\frac{p}{p-p'}\right)^{p/p'}\sum_{n=1}^{\infty}\frac{1}{n^{2p}}\left(T^{p/2}\Gamma\left(\frac{p}{2}+1\right)\right)^n$$
$$<\infty.$$

The last inequality "$<\infty$" follows from Lemma 4. Thus we have

$$\sup_{0\le t\le T}\left(\sum_{n=1}^{\infty}|G_n(Z_t)|\right)^p<\infty \quad \text{a.s.},$$

which implies

$$P\left(\sum_{n=1}^{\infty}|G_n(Z_t)|<\infty \ \ 0\le {}^{\forall}t\le T\right)=1,$$

or equivalently, $\sigma_A \geq T$ a.s. Since $0<T<e^{\gamma}$ is arbitrary, we finally obtain $\sigma_A \geq e^{\gamma}$ a.s.

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