Remark on the Range Inclusions of Toeplitz and Hankel Operators*)

By Yuka SONE and Takashi YOSHINO

Mathematical Institute, Tôhoku University

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Abstract: In this paper, we study the equivalent relations between range inclusions and symbols of Toeplitz and Hankel operators, and give some applications.

Let μ be the normalized Lebesgue measure on the Borel sets of the unit circle in the complex plane $\boldsymbol{C}$. If $e_n(z) = z^n$ for $|z| = 1$ and $n = 0, \pm 1, \pm 2, \cdots$, then the bounded measurable functions e_n constitute an orthonormal basis for $L^2 = L^2(\mu)$. And the functions e_n, $n = 0,1,2,\cdots$ constitute the orthonormal basis for H^2.

For φ in L^∞, the Laurent operator L_φ is the multiplication operator on L^2 given by $L_\varphi f = \varphi f$ for $f \in L^2$. And the Toeplitz operator T_φ is the operator on H^2 given by $T_\varphi f = PL_\varphi f$ for $f \in H^2$, where P is the orthogonal projection from L^2 onto H^2. The Hankel operator H_φ is the operator on H^2 given by $H_\varphi f = J(I-P)L_\varphi f$ for $f \in H^2$, where J is the unitary operator on L^2 given by $J(z^{-n}) = z^{n-1}$, $n = 0, \pm 1, \pm 2, \cdots$.

The following results are well known, but, for convenience's sake we state here them without proof.

Proposition 1. T_φ has the following properties.

(1) $T_z^* T_\varphi T_z = T_\varphi$, where T_z^* denotes the adjoint operator of T_z.

(2) $T_\varphi^* = T_{\bar\varphi}$, where the bar denotes the complex conjugate.

(3) $T_{\alpha\varphi+\beta\psi} = \alpha T_\varphi + \beta T_\psi$, $\alpha, \beta \in \boldsymbol{C}$.

(4) $T_\varphi = O$ if and only if $\varphi = o$.

(5) $\| T_\varphi \| = \| \varphi \|_\infty$.

We denote the set of all bounded linear operators on a Hilbert space $\mathscr{H}$ by $\mathscr{B}(\mathscr{H})$.

Proposition 2 ([1]). $A \in \mathscr{B}(H^2)$ is a Toeplitz operator if and only if $T_z^* A T_z = A$. And, in particular, $A \in \mathscr{B}(H^2)$ is analytic Toeplitz operator (i.e., $A = T_\varphi$ for some $\varphi \in H^\infty$) if and only if $T_z A = A T_z$.

Proposition 3. H_φ has the following properties.

(1) $T_z^* H_\varphi = H_\varphi T_z$.
(Hence $\mathcal{N}_{H_\varphi} = \{x \in H^2 ; H_\varphi x = o\}$ is invariant under T_z and $\mathcal{N}_{H_\varphi} = \{o\}$ or $\mathcal{N}_{H_\varphi} = T_q H^2$, where q is inner).

(2) $H_\varphi^* = H_{\varphi^*}$, where $\varphi^*(z) = \overline{\varphi(\bar z)}$.

(3) $H_{\alpha\varphi+\beta\psi} = \alpha H_\varphi + \beta H_\psi$, $\alpha, \beta \in \boldsymbol{C}$.

(4) $H_\varphi = O$ if and only if $(I-P)\varphi = o$ (i.e., $\varphi \in H^\infty$).

(5) $\| H_\varphi \| = \inf\{\| \varphi + \psi \|_\infty ; \psi \in H^\infty\}$.

Proposition 4. $A \in \mathscr{B}(H^2)$ is a Hankel operator if and only if $T_z^* A = A T_z$. Moreover we can choose the symbol $\varphi \in L^\infty$ of $A = H_\varphi$ such as $\| A \| = \| \varphi \|_\infty$.

The following relations between Toeplitz and Hankel operators are known.

Proposition 5 ([5]). $H_\varphi^* H_\varphi = T_{\bar\varphi\varphi} - T_{\bar\varphi} T_\varphi$. And, for any $\psi \in H^\infty$, $H_\varphi T_\psi = H_{\varphi\psi}$ and $T_\psi^* H_\varphi = H_\varphi T_{\psi^*}$.

Concerning the range inclusions of Toeplitz and Hankel operators, the following results are known.

Proposition 6 ([6]). If φ and ψ are in H^∞, then $T_\varphi H^2 \subseteq T_\psi H^2$ if and only if there exists a $g \in H^\infty$ uniquely such that $T_\varphi = T_\psi T_g = T_{\psi g}$. And then $\varphi = \psi g$. Particularly, if φ and ψ are inner, then g is also inner.

Proposition 7 ([5]). The following assertions are equivalent.

(1) $H_{\varphi_1} H^2 \subseteq H_{\varphi_2} H^2$.

(2) $H_{\varphi_1} H_{\varphi_1}^* \leq \lambda^2 H_{\varphi_2} H_{\varphi_2}^*$ for some $\lambda \geq 0$.

(3) There exists a function $h \in H^\infty$ such that $\| h \|_\infty \leq \lambda$ for some $\lambda \geq 0$ and that $H_{\varphi_1} = H_{\varphi_2} T_h = H_{\varphi_2 h}$.

(4) There exists a function $h \in H^\infty$ such that $\| h \|_\infty \leq \lambda$ for some $\lambda \geq 0$ and that $\varphi_1 - \varphi_2 h \in H^\infty$.

Proposition 8 ([3]). $T_\varphi^* H^2 \subseteq H_\psi^* H^2$ if and

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only if $\varphi = o$.

Proposition 9 ([3]). The following assertions are equivalent.

(1) $H_\varphi^* H^2 \subset T_\varphi^* H^2$.

(2) P is bounded below on $[L_\varphi H^2]^{\sim L^2} \neq \{o\}$, where $[L_\varphi H^2]^{\sim L^2}$ denotes the closure of $L_\varphi H^2$ in L^2.

In Proposition 7, $H_{\varphi_1} = H_{\varphi_2} T_h$ for $h \in H^\infty$ implies that $H_{\varphi_1}^* = T_h^* H_{\varphi_2}^*$ and $H_{\varphi_1}^* H^2 \subseteq T_h^* H^2$. And concerning this and Proposition 9, we have the following.

Theorem 1. For $\phi \in H^\infty$, $H_\varphi H^2 \subseteq T_\phi^* H^2$ if and only if there exists a function $u \in L^\infty$ such that $H_\varphi = T_\phi^* H_u$.

To prove this theorem, we need the following two lemmas.

Lemma 1 ([2]). For $A, B \in \mathscr{B}(\mathscr{H})$, the following assertions are equivalent.

(1) $A\mathscr{H} \subseteq B\mathscr{H}$.

(2) $AA^* \leq \lambda^2 BB^*$ for some $\lambda \geq 0$.

(3) There exists a $C \in \mathscr{B}(\mathscr{H})$ such that $A = BC$.

In particular, there exists a $C \in \mathscr{B}(\mathscr{H})$ uniquely such that

(a) $\| C \|^2 = \inf\{\mu : AA^* \leq \mu BB^*\}$

(b) $\mathscr{N}_A = \mathscr{N}_C$ and (c) $C\mathscr{H} \subseteq [B^*\mathscr{H}]^\sim$.

Lemma 2. For any non-zero $f \in H^2$, there exist an inner function φ and an outer function h uniquely such that $f = \varphi h$.

Proof of Theorem 1. $H_\varphi H^2 = T_\phi^* H_u H^2 \subseteq T_\phi^* H^2$.

Conversely if $H_\varphi H^2 \subseteq T_\phi^* H^2$, then we may assume $\phi \neq o$ because, in the case where $\phi = o$, we have $H_\varphi = O$ and the assertion is clear. And then, by Lemma 2, $\phi = gh$ where g is inner and h is outer. Since

$$T_g H^2 = T_g T_g^* T_g H^2 \subseteq T_g T_g^* H^2,$$
$$H^2 = T_g^* T_g H^2 \subseteq T_g^*(T_g T_g^* H^2) = T_g^* H^2 \subseteq H^2$$

and $T_g^* H^2 = H^2$ and hence $T_\phi^* H^2 = T_h^* T_g^* H^2 = T_h^* H^2$. Hence, by the assumption, $H_\varphi H^2 \subseteq T_h^* H^2$ and, by Lemma 1, there exists an $A \in \mathscr{B}(H^2)$ uniquely such that $H_\varphi = T_h^* A$ and that

(a) $\| A \|^2 = \inf\{\mu : H_\varphi H_\varphi^* \leq \mu T_h^* T_h\}$

(b) $\mathscr{N}_{H_\varphi} = \mathscr{N}_A$ and (c) $AH^2 \subseteq [T_h H^2]^{\sim L^2}$.

Then, $T_h^* T_z^* A = T_z^* T_h^* A = T_z^* H_\varphi = H_\varphi T_z = T_h^* A T_z$ by Propositions 2 and 3. Since h is outer, $H^2 = \vee\{z^n h : n = 0,1,2,\cdots\} = [T_h H^2]^{\sim L^2}$ and $\mathscr{N}_{T_h^*} = \{o\}$ and hence $T_z^* A = A T_z$. Therefore, by Proposition 4, A is a Hankel operator. i.e., $A = H_v$ for some $v \in L^\infty$. And then, by Proposition 5,

$$H_\varphi = T_h^* H_v = H_v T_{h^*} = H_{vh^*} = H_{v\overline{g^*}g^*h^*} = H_{u\phi^*} = H_u T_{\phi^*} = T_\phi^* H_u,$$ where $u = v\,\overline{g^*} \in L^\infty$.

Concerning Proposition 8, we have the following.

Theorem 2. If $[T_\varphi H^2]^{\sim L^2} \subseteq [H_\phi H^2]^{\sim L^2} \neq H^2$, then $\varphi = o$.

Proof. If $[T_\varphi H^2]^{\sim L^2} \subseteq [H_\phi H^2]^{\sim L^2} \neq H^2$, then $\{o\} \neq \mathscr{N}_{H_\phi^*} \subseteq \mathscr{N}_{T_\varphi^*}$ and, by Proposition 3, $\mathscr{N}_{H_\phi^*} = T_g H^2$ for some inner function g and hence $T_\varphi^* T_g H^2 = \{o\}$. i.e., $T_{\bar{\varphi} g} = T_\varphi^* T_g = O$ and hence $\bar{\varphi} g = o$ by Proposition 1. Since g is non-zero analytic, $\varphi = o$ by F. and M. Riesz theorem (i.e., a non-zero analytic function can not vanish on a set of positive measure).

As a special case of Proposition 7, we have the following.

Theorem 3. H_φ is hyponormal (i.e., $H_\varphi H_\varphi^* \leq H_\varphi^* H_\varphi$) if and only if $H_\varphi = H_\varphi^* T_h$ (i.e., $\varphi - \varphi^* h \in H^\infty$) for some $h \in H^\infty$ such as $\| h \|_\infty \leq 1$. And, in this case, $H_\varphi T_z$ is also hyponormal.

Proof. Since H_φ is hyponormal if and only if $H_\varphi H_\varphi^* \leq H_\varphi^* H_\varphi = H_{\varphi^*} H_{\varphi^*}^*$ by Proposition 3, it is equivalent that there exists a function $h \in H^\infty$ such as $\| h \|_\infty \leq 1$ and $H_\varphi = H_\varphi^* T_h$ by Proposition 7. And, by Propositions 3 and 5, we have

$$H_{\varphi z} = H_\varphi T_z = H_\varphi^* T_h T_z = H_\varphi^* T_z T_h = T_z^* H_\varphi^* T_h = (H_\varphi T_z)^* T_h = H_{\varphi z}^* T_h$$

and hence $H_\varphi T_z$ is also hyponormal.

By [1], it is known that Toeplitz operator T_φ is normal if and only if $T_\varphi = \lambda T_\phi + \mu I$ for some $\lambda, \mu \in C$ and ϕ such as $\bar{\phi} = \phi$ (i.e., T_ϕ is Hermitian). In the case of Hankel operator, as an application of Theorem 3, we have the following.

Theorem 4. The normal Hankel operator is only a scalar multiple of a Hermitian Hankel operator.

Proof. Clearly a scalar multiple of a Hermitian Hankel operator is a normal Hankel operator.

Conversely if H_φ is normal, then $H_\varphi H_\varphi^* = H_\varphi^* H_\varphi$ and, by Theorem 3, there exist functions g and h in H^∞ such that $\| g \|_\infty \leq 1$, $\| h \|_\infty \leq 1$, $H_\varphi = H_{\varphi^*} T_g = H_{\varphi^* g}$ and $H_{\varphi^*} = H_\varphi T_h = H_{\varphi h}$. And then $H_\varphi = H_{\varphi^*} T_g = H_\varphi T_h T_g = H_\varphi T_{hg}$ and $(T_{gh}^* - I) H_\varphi^* = O$. Since $\| T_{gh} \| \leq \| g \|_\infty \| h \|_\infty \leq 1$, $T_{gh}^* u = u$ if and only if $T_{gh} u = u$ and since $\sigma_p(T_{gh}) \cap \sigma_p(T_{gh}^*) = \emptyset$ whenever gh is non-constant by [4; Theorem 7], $gh = 1$ or

$H_\varphi{}^* H^2 = \{o\}$ (i.e., $H_\varphi{}^* = O$). Clearly $H_\varphi{}^* = O$ is Hermitian. In the case where $gh = 1$, since $1 = |gh| = |g||h|$ and since $\|g\|_\infty, \|h\|_\infty \leq 1$, $|g| = |h| = 1$ a.e. (i.e., g and h are inner) and hence T_g and T_h are isometries. Since $T_g T_h = T_{gh} = I$, T_g and T_h are invertible and T_g and T_h are unitary and hence g and h are constant functions of absolute value 1. Then $g = \bar{h} = e^{i\theta_0}1$ for some $\theta_0 \in [0, 2\pi)$ and

$$H_\varphi = H_{\varphi^* g} = H_{e^{i\theta_0}\varphi^*} = e^{i\theta_0} H_{\varphi^*} = e^{i\theta_0} H_\varphi{}^*$$

by Proposition 3 and hence, for any $r > 0$,

$$H_{\frac{1}{r}e^{-\frac{i\theta_0}{2}}\varphi} = \frac{1}{r} e^{-\frac{i\theta_0}{2}} H_\varphi = \frac{1}{r} e^{\frac{i\theta_0}{2}} H_\varphi{}^*$$
$$= \left(\frac{1}{r} e^{-\frac{i\theta_0}{2}} H_\varphi\right)^* = H_{\frac{1}{r}e^{-\frac{i\theta_0}{2}}\varphi}{}^*.$$

Therefore $H_\varphi = re^{\frac{i\theta_0}{2}} H_\phi$, where $H_\phi = H_{\frac{1}{r}e^{-\frac{i\theta_0}{2}}\varphi}$ is Hermitian.

By Proposition 1, Toeplitz operator T_φ is Hermitian (i.e., $T_\varphi{}^* = T_\varphi$) if and only if $\bar{\varphi} = \varphi$. Hermitian Hankel operator is characterized as follows.

Theorem 5. $H_\varphi{}^* = H_\varphi$ if and only if, for

$$\varphi(z) = \sum_{n=-\infty}^{\infty} \lambda_n z^n,$$

$\lambda_n \in \boldsymbol{R}$ for all $n = -1, -2, \cdots$.

Proof. Since $H_\varphi{}^* = H_\varphi$ if and only if $\varphi^* - \varphi \in H^\infty$ by Proposition 3 and since

$$\varphi^*(z) - \varphi(z)$$
$$= \overline{\varphi(\bar{z})} - \varphi(z) = \overline{\sum_{n=-\infty}^{\infty} \lambda_n \bar{z}^n} - \sum_{n=-\infty}^{\infty} \lambda_n z^n$$
$$= \sum_{n=-\infty}^{\infty} \overline{\lambda_n} z^n - \sum_{n=-\infty}^{\infty} \lambda_n z^n = \sum_{n=-\infty}^{\infty} (\overline{\lambda_n} - \lambda_n) z^n,$$

$H_\varphi{}^* = H_\varphi$ if and only if $\overline{\lambda_n} - \lambda_n = 0$ for all $n = -1, -2, \cdots$.

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