8. *On Rational Approximations to Linear Forms in Values of G-functions*

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C. L. Siegel [8] defined E-functions and G-functions. They are solutions of linear differential equations which are expressed as Taylor series $\sum_{m=0}^{\infty} a_m x^m$ with coefficients in an algebraic number field.

E-functions satisfy

(1) for any $\varepsilon > 0$, the absolute values of $m!a_m$ and its conjugates do not exceed $Cm^{\varepsilon m}$,

(2) there is a sequence of common denominator d_m for $a_0, a_1, 2!a_2, \dots, m!a_m$ which does not exceed $Cm^{\varepsilon m}$.

And G-functions satisfy

(3) the absolute values of a_m and its conjugates do not exceed C^m,

(4) there is a sequence of common denominator d_m for $a_0, a_1, a_2, \dots, a_m$ which does not exceed C^m.

(C is a sufficiently large positive constant which is independent of m.)

Siegel utilized E-functions to obtain some results in the theory of transcendental numbers and suggested that G-functions will be also useful for similar purposes. The theory of these functions has been developed by different authors. In particular, Shidlovskii [7] proved the transcendency of the values in E-functions using the classical Padé approximations. The purpose of this paper is to show that we can use the method of [7] to G-functions in making more precise the definition of Padé approximations to obtain the best possible bounds of irrationality measure for linear independence in values of G-functions in some cases.

We assume $f_i^{(j)}(x) \in \mathbf{Q}[[x]]$ have non zero radii of convergence at $x = 0$.

We have the following.

Theorem A. *Let $f_i(x)$ $(i = 1, \dots, n)$ be a non zero solution of a scalar linear differential equation of the order m_i over $\mathbf{Q}(x)$:*

$$\left(\frac{d}{dx}\right)^{m_i} f_i(x) + a_{m_i-1}^{(i)}(x)\left(\frac{d}{dx}\right)^{m_i-1} f_i(x) + \cdots + a_0^{(i)}(x) f_i(x) = 0,$$

where $a_j^{(i)}(x) \in \mathbf{Q}(x)$ $(i = 1, \dots, n; j = 0, \dots, m_i - 1)$. Put $f_i^{(j)}(x) := \left(\frac{d}{dx}\right)^j f_i(x)$ and $m := \sum_{i=1}^n m_i$. Suppose $f_i^{(j)}(x)$ $(i = 1, \dots, n; j = 0, \dots, m_i - 1)$ are linearly independent over $\mathbf{Q}(x)$. Let ε_0 be fixed in $\frac{1}{2} > \varepsilon_0 > 0$. Then there are effective constants C_1, depending only on $f_i^{(j)}(x)$ $(i = 1, \dots, n; j = 0, \dots, m_i - 1)$, ε_0, m and C_2, depending only on $f_i^{(j)}(x)$ $(i = 1, \dots, n; j = 0, \dots, m_i - 1)$, ε_0, m, r such that the following holds.

Let r be any rational number

$$r = \frac{a}{b} \neq 0 \ \ a, b \in \boldsymbol{Z} \ with \ |b|^{\varepsilon_0} > C_1 |a|^{2m(m+1)}.$$

Then for arbitrary integers $H_i^{(j)}$ $(i = 1, \ldots, n\,;\, j = 0, \ldots, m_i - 1)$ *satisfying* $H := \max_{i=1,\ldots,n}(|H_i|) > C_2$ *and* $H_i := \max_{j=0,\ldots,m_i-1}(|H_i^{(j)}|, 1)$, *we have*

$$\left|\sum_{i=1}^{n}\sum_{j=0}^{m_i-1} H_i^{(j)} f_i^{(j)}(r)\right| > \frac{H^{1-\varepsilon_0}}{H_1^{m_1}\cdots H_n^{m_n}}.$$

Theorem A gives the best possible bounds in values of G-functions in the case of $m_1 = \cdots = m_n = 1$.

Under the conditions of Theorem A, we assume that $f_i^{(j)}(x)$ $(i = 1, \ldots, n\,;\, j = 0, \ldots, m_i - 1)$ is a system of G-functions satisfying the system of the first order differential equation over $\boldsymbol{Q}(x)$:

$$\text{[Eq.1]} \qquad \frac{d}{dx}\begin{pmatrix} f_1^{(0)}(x) \\ \vdots \\ f_1^{(m_1-1)}(x) \\ \vdots \\ f_n^{(0)}(x) \\ \vdots \\ f_n^{(m_n-1)}(x) \end{pmatrix} = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{pmatrix}\begin{pmatrix} f_1^{(0)}(x) \\ \vdots \\ f_1^{(m_1-1)}(x) \\ \vdots \\ f_n^{(0)}(x) \\ \vdots \\ f_n^{(m_n-1)}(x) \end{pmatrix}$$

for $A_i \in M_{m_i}(\boldsymbol{Q}(x)) : i = 1, \ldots, n$.

For the proof of Theorem A we use Padé approximations for G-functions which are essentially the same as those Shidlovskii [7] used for E-functions. But in order to sharpen the bound, we shall use the following definitions.

Definition of Padé approximations. Let $f_i^{(j)}(x)$ $(i = 1, \ldots, n\,;\, j = 0, \ldots, m_i - 1) \in \boldsymbol{C}[[x]]$. Let $\varepsilon > 0$ be a sufficiently small and fixed real number. For positive parameters: $D, D_i (i = 1, \ldots, n) \in \boldsymbol{Z}$, $T \in \boldsymbol{R}$, suppose that $P_i^{(j)}(x)$ $(i = 1, \ldots, n\,;\, j = 0, \ldots, m_i - 1)$ are polynomials which satisfy deg $P_i^{(j)}(x) \leq D$, $\mathrm{ord}_{x=0} P_i^{(j)}(x) \geq D - D_i$ and further

$$R(x) := \sum_{i=1}^{n}\sum_{j=0}^{m_i-1} P_i^{(j)}(x) f_i^{(j)}(x)$$

satisfies $\mathrm{ord}_{x=0} R(x) \geq T$ and $R(x) \not\equiv 0$. Then we call $P_i^{(j)}(x)$ $(i = 1, \ldots, n\,;\, j = 0, \ldots, m_i - 1)$ as *Padé approximants* and $R(x)$ as *the remainder function* in the Padé approximation problem with the parameters (T, D, D_i) for $f_i^{(j)}(x)$ $(i = 1, \ldots, n\,;\, j = 0, \ldots, m_i - 1)$. Moreover we assume $D := \max_{i=1,\ldots,n} D_i$ and

$$D_i \geq 2\varepsilon D, \ T \geq \left[\sum_{i=1}^{n} m_i D_i - \varepsilon D\right].$$

If we take the absolute values of coefficients of the linear form of Theorem A, H_i, is larger, we should make D_i of Padé approximants larger and vise versa.

Our proof of Theorem A requires a sequence of many Lemmas, so we will describe only a key Lemma, which is an improvement of Shidlovskii's one.

Now we define m-tuples of polynomials such that

$$\bar{p}^{[0]} := {}^t(P_1^{(0)}(x) \ldots P_1^{(m_1-1)}(x) \ldots P_n^{(0)}(x) \ldots P_n^{(m_n-1)}(x)),$$

and recursively for $k = 0,1,\ldots,$

$$\bar{p}^{[k+1]}(x) := d(x)\Big(\frac{d}{dx} I + {}^tA\Big)\bar{p}^{[k]}(x),$$

where A is the coefficient matrix in the differential equation [Eq.1] and $d(x)$ = (the common denominator of entries of A) $\in \boldsymbol{Z}[x]$.

Lemma [cf. 7, ch. 3, lemma 10]. *Let* $f_i^{(j)}(x)$ $(i = 1,\ldots,n\,;\, j = 0,\ldots,m_i - 1)$ *satisfy the differential equation* [Eq.1] *and be linearly independent over* $\boldsymbol{C}(x)$. *Then for any number* x_0 *such that* $x_0\, d(x_0) \neq 0$, *we have*

$$rank(\bar{p}^{[k]}(x_0))_{k=0,\ldots,[2\varepsilon D]} = m$$

for any large D.

We use the property of the coefficient matrix of diagonal blocks in the differential equation [Eq.1] to show Theorem A. However we need not adhere to this type of the coefficient matrix. We take another differential equation over $\boldsymbol{Q}(x)$ substituted for [Eq.1] as following:

[Eq.2] $$\frac{d}{dx}\begin{pmatrix} f_1^{(0)}(x) \\ \vdots \\ f_1^{(m_1-1)}(x) \\ \vdots \\ f_n^{(0)}(x) \\ \vdots \\ f_n^{(m_n-1)}(x) \end{pmatrix} = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ * & & & A_n \end{pmatrix}\begin{pmatrix} f_1^{(0)}(x) \\ \vdots \\ f_1^{(m_1-1)}(x) \\ \vdots \\ f_n^{(0)}(x) \\ \vdots \\ f_n^{(m_n-1)}(x) \end{pmatrix}$$

for $A_i \in M_{m_i}(\boldsymbol{Q}(x)) : i = 1,\ldots,n$.

Theorem B. *Let* $f_i^{(j)}(x)$ $(i = 1,\ldots,n\,;\, j = 0,\ldots,m_i - 1)$ *be a non zero solution of the linear differential equation over* $\boldsymbol{Q}(x)$ [Eq.2]. *Put* $m := \sum_{i=1}^n m_i$. *Suppose* $f_i^{(j)}(x)$ $(i = 1,\ldots,n\,;\, j = 0,\ldots,m_i - 1)$ *are linearly independent over* $\boldsymbol{Q}(x)$. *Let* ε_0 *be fixed in* $\frac{1}{2} > \varepsilon_0 > 0$. *Then there are effective constants* C_3, *depending only on* $f_i^{(j)}(x)$ $(i = 1,\ldots,n\,;\, j = 0,\ldots,m_i - 1)$, ε_0, m *and* C_4, *depending only on* $f_i^{(j)}(x)$ $(i = 1,\ldots,n\,;\, j = 0,\ldots,m_i - 1)$, ε_0, m, r *such that the following holds.*

Let r *be any rational number*

$$r = \frac{a}{b} \neq 0 \quad a,\, b \in \boldsymbol{Z} \text{ with } |\, b\,|^{\varepsilon_0} > C_3\,|\, a\,|^{2m(m+1)}.$$

Then for arbitrary integers $H_i^{(j)}$ $(i = 1,\ldots,n\,;\, j = 0,\ldots,m_i - 1)$ *satisfying* $H_1 > C_4$ *and* $H_1 \geq H_2 \geq \cdots \geq H_n > 0$, *we have*

$$|\sum_{i=1}^{n}\sum_{j=0}^{m_i-1} H_i^{(j)} f_i^{(j)}(r)\,| > \frac{H_1^{1-\varepsilon_0}}{H_1^{m_1}\cdots H_n^{m_n}},$$

where $H_i := \max_{j=0,\ldots m_i-1}(|\, H_i^{(j)}\,|) \neq 0$.

Theorem B applies some concrete G-functions such as the logarithm and polygarithms.

The details will appear elsewhere.

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