29. *On Inner Amenability of Clifford Semigroups*

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§0. Introduction. The inner amenability of groups is investigated by many authors e.g., [4]-[8], [15] and [16]. In this paper we shall introduce the inner amenability for Clifford semigroups and show various necessary and sufficient conditions for Clifford semigroups to be inner amenable.

Throughout this paper S is a Clifford semigroup, i.e., S is an inverse semigroup such that the set E_S of all idempotent elements in S is contained in the center $Z(S)$ of S (cf. [9]). For any $s \in S$ there corresponds a unique $s^* \in S$, the inverse of s, such that $ss^*s = s$ and $s^*ss^* = s^*$. Since ss^*, $s^*s \in E_S \subseteq Z(S)$, we can define the inner endomorphism $c(s)$ on S by $c(s)t = sts^* (t \in S)$.

For any space X, let $B(X)$ be the Banach space of all bounded functions on X with the supremum norm. A mean ϕ on X is a positive linear functional on $B(X)$ such that $\phi(1_X) = 1$, where in general 1_Y is the characteristic function of any $Y \subseteq X$. For brevity we write $\phi(Y) = \phi(1_Y)$.

For $f \in B(S)$ and $s \in S$ we define $c(s)f \in B(s)$ by $c(s)f(a) = f(sas^*)$, $a \in S$. A mean ϕ on S is said to be *inner invariant* if $\phi(c(s)f) = \phi(f)$ for any $s \in S$ and $f \in B(S)$. A subset $V \subseteq S$ is said to be *inner invariant* if $c(s)^{-1}V = V$ for any $s \in S$, where $c(s)^{-1}V = \{a \in S ; sas^* \in V\}$. S is said to be *inner amenable on an inner invariant subset* V if there exists an inner invariant mean ϕ on S such that $\phi(V) = 1$.

In S we can introduce a congruence relation ρ by $s_1(\rho)s_2$, $s_1, s_2 \in S$ if and only if $s_1e = s_2e$ for some $e \in E_S$ (cf. [9]). Then the quotient semigroup $G_S = S/\rho$ becomes a group. We denote also by ρ the canonical homomorphism of S onto G_S. Then $\rho(s^*) = \rho(s)^{-1}$, $s \in S$, Evidently ρ transforms the inner endomorphism $c(s)$ to the inner automorphism $c(\rho(s))$ on G_S induced by $\rho(s)$. We set $Z_S = \rho^{-1}(Z(G_S))$, where $Z(G_S)$ is the center of G_S.

In the section 1 we establish the relation between the inner amenability of S and G_S. The section 2 gives various conditions for S to be inner amenable on any inner invariant subset of S, which are derived from author's papers [10]-[12] and[13]. Especially we show the fixed point theorem corresponding to the inner amenability of S. In section 3 we give some conditions for S to be inner amenable on $S - Z_S$, the complement of Z_S in S, which are generalizations of the main result in Paschke [8].

Throughout this paper V is an inner invariant subset of S and $W = \rho(V)$, which is also an inner invariant subset of G_S, i.e., $W = \rho(s)W\rho(s)^{-1}$ for any $s \in S$. We note that $\rho^{-1}(W') \subseteq S$ is inner invariant for any inner invariant $W' \subseteq G_S$.

§1. Amenability of S and G_S. By $M(S)$ [resp. $IM(S)$] we denote the space of all means [resp. inner invariant means] on S, and set $M(S, A) = \{\phi \in M(S)\,;\, \phi(A) = 1\}$ and $IM(S, A) = M(S, A) \cap IM(S)$ for any $A \subseteq S$. Since E_S is a commutative subsemigroup of S, there exists a translation invariant mean ψ on E_S (cf. [1]), i.e., $\psi(h_e) = \psi(h)$ for any $h \in B(E_S)$ and $e \in E_S$, where $h_e(t) = h(te)\,(t \in E_S)$. For any $f \in B(S)$, we define $f^\wedge \in B(S)$ by $f^\wedge(s) = \psi(sf)$, $s \in S$, where $sf \in B(E_S)$ is given by $sf(e) = f(se)$, $e \in E_S$. Since $f^\wedge(se) = f^\wedge(s)$ for any $s \in S$ and $e \in E_S$ by the translation invariance of ψ, $f^\wedge$ is consider as a function on G_S. So for any $f \in B(S)$ we define $\rho f \in B(G_S)$ by $\rho f(\rho(s)) = f^\wedge(s)$, $s \in S$. On the other hand, for any $h \in B(G_S)$ we define $\rho^* h \in B(S)$ by $\rho^* h(s) = h(\rho(s))$, $s \in S$. The next is easily seen.

Lemma 1. (1) $\rho(1_V) = 1_W$ *and* $\rho(c(s)f) = c(\rho(s))\rho f$ *for* $f \in B(S)$ *and* $s \in S$.
(2) $\rho^*(1_W) = 1_V$ *and* $\rho^*(c(\rho(s))h) = c(s)\rho^* h$ *for any* $h \in B(G_S)$ *and* $s \in S$.

Let $\phi^\wedge \in IM(G_S, W)$ and define $\phi \in M(S)$ by $\phi(f) = \phi^\wedge(\rho f)$, $f \in B(S)$. Then we see from Lemma 1(1) that $\phi \in IM(S, V)$. Conversely for any $\phi \in IM(S, V)$, we define $\phi^\wedge \in M(G_S)$ by $\phi^\wedge(h) = \phi(\rho^* h)$, $h \in B(G_S)$. Then $\phi^\wedge \in IM(G_S, W)$. Therefore we have

Theorem 1. *S is inner amenable on V if and only if G_S is inner amenable on W.*

The above is analogous to the fact that S is amenable if and only if so is G_S (cf. [3]). Let $E = \rho^{-1}(\{\mathbf{e}\})$, where $\mathbf{e}$ is the identity in G_S. Evidently G_S is inner amenable on $\{\mathbf{e}\}$ and $Z(G_S)$ respectively. So from Theorem 1 we have

Corollary 1. *S is inner amenable on E and Z_S respectively.*

For any $s \in S$ we define a map $c^*(s)$ on $M(S)$ by $c^*(s)\phi(f) = \phi(c(s)f)$, $f \in B(S)$. Then the map c^* is a homomorphism of S to the semigroup of all continuous affine maps on $M(S)$. Moreover we see that $c^*(s)\phi \in M(S, V)$ for any $\phi \in M(S, V)$ and $s \in S$. $M(S)$ and $M(S, V)$ are w^*-compact convex subsets of the dual space $B(S)^*$. From these facts and Day's fixed point theorem (cf. [1], [2]) we have

Theorem 2. *If S is amenable, then S is inner amenable on V.*

Let $\phi \in IM(S)$ and $\alpha = \phi(V)$. We suppose $0 < \alpha < 1$ and define the means ϕ_1 and ϕ_2 on S by $\phi_1(f) = \alpha^{-1}\phi(1_V f)$ and $\phi_2(f) = (1-\alpha)^{-1}\phi(1_{S-V} f)$, $f \in B(S)$, respectively. Then $\phi_1 \in IM(S, V)$ and $\phi_2 \in IM(S, S - V)$. So we have

Theorem 3. *Let $\phi \in IM(S)$ and $\alpha = \phi(V)$. Then S is inner amenable on V [resp. $S - V$] if $\alpha > 0$ [resp. $\alpha < 1$].*

From Corollary 1 and the proof of Theorem 3, we see that every $\phi \in IM(S)$ is expressed in the convex linear combination of $\phi_1 \in IM(S, Z_S)$ and $\phi_2 \in IM(S, S - Z_S)$.

§2. Characterizations of inner amenability. In order to state the main theorem we begin with some notations. For any $p \in [1, \infty)$, let $L_p(S)$ be the

usual Banach space of functions on S with respect to the counting measure on S. We set:
$M_p(S) = \{h \in L_p(S)\ ;\ h \geqq 0$ and $\|h\|_p = 1\}$, $F_p(S) = \{h \in M_p(S)\ ;\ \mathrm{supp}(h)$ is finite$\}$, and for any $A \subseteq S$,
$L_p(S, A) = \{h \in L_p(S)\ ;\ \mathrm{supp}\,(h) \subseteq A\}$, $M_p(S, A) = L_p(S, A) \cap M_p(S)$, $F_p(S, A) = M_p(S, A) \cap F_p(S)$.
Every $h \in M_1(S)$ is identified with a mean on S defined by $h(f) = \sum\{h(s)f(s)\ ;\ s \in S\}$. So $M_1(S)$ [resp. $M_1(S, A)$] is regarded as a subset of $M(S)$ [resp. $M(S, A)$]. We note that $F_1(S)$ [resp. $F_1(S, A)$] is weakly* dense in $M(S)$ [resp. $M(S, A)$].

For $s \in S$ we put $R(s) = \{a \in S\ ;\ ss^*a = s^*sa = a\}$. Since ss^*, $s^*s \in E_S \subseteq Z(S)$, $R(s)$ is an ideal of S and $R(s) = c(s)S = c(s)R(s)$. $c(s)$ and $c(s^*)$ are bijective on $R(s)$, and $c(s^*) = c(s)^{-1}$ on $R(s)$. Let $P(S)$ be the family of all finite subsets of S. For $K \in P(S)$ we set $R(K) = \cap \{R(s)\ ;\ s \in K\}$, which is also an ideal of S. We see that $\rho(R(K)) = G_S$ and that $\phi(V \cap R(K)) = 1$ for any $\phi \in M(S, V)$ such that $c^*(s)\phi = \phi$ for all $s \in K$.

Let us fix $p \in [1, \infty)$. For $s \in S$ we define an linear operator $c_p(s)$ on $L_p(S)$ by $c_p(s)h(a) = h(s^*as)$ if $a \in R(s)$ and $c_p(s)h(a) = 0$ if $a \notin R(s)$ for $h \in L_p(S)$. Then $c_1(s)h = c^*(s)h$ for $h \in M_1(S, R(s))$, and $c_p(s)$ induces an isometry on $L_p(S, R(s))$. Moreover we see that $c_p(s)c_p(t) = c_p(st)$ for s, $t \in S$ from the relation $R(st) = R(s) \cap R(t)$.

Now we consider the following condition $(P)_p$: *For any $K \in P(S)$ and $\varepsilon > 0$ there exists $h \in F_p(S, R(K))$ such that $\|c_p(s)h - h\|_p < \varepsilon$ for all $s \in K$.* By the same method as in [14] we see that $(P)_p \Leftrightarrow (P)_q$ for any p, $q \in [1, \infty)$. Under these notations, the next theorem is derived from author's papers [10], [11] and [13].

Theorem 4. *The following conditions are mutually equivalent.*

(1) *S is inner amenable on V.*

(2) *There exists a net $\{\phi_\alpha\}$ in $F_1(S, V)$ such that*
$$w^*\text{-}\lim_\alpha (c^*(s)\phi_\alpha - \phi_\alpha) = 0 \textit{ for any } s \in S.$$
(3) *There exists a net $\{\phi_\alpha\}$ in $F_1(S, V)$ such that*
$$\lim_\alpha \|c^*(s)\phi_\alpha - \phi_\alpha\|_1 = 0 \textit{ for any } s \in S.$$
(4) *$(P)_1$ holds.*

(5) *$(P)_p$ holds for some $p \in (1, \infty)$.*

(6) *For any $K \in P(S)$ and $\varepsilon > 0$ there exists a finite subset $A \subseteq S$ such that*
$$|A - sAs^*| = |sAs^* - A| < \varepsilon |A| \textit{ for all } s \in K,$$
where $|B|$ denotes the cardinality of any finite set B.

We show the fixed point property corresponding to the inner amenability of S. *A compact affine conjugate action of S* is a quadruplet $\{Q, T, \pi, \tau\}$ with the following properties (a)–(c):

(a) *Q is a compact convex subset of a locally convex topological linear space T,*

(b) *π is a homomorphism of S to the semigroup of all continuous affine maps on Q,*

(c) *τ is a map of S to Q such that $\tau(sts^*) = \pi(s)\tau(t)$ for any s, $t \in S$.*

We define a map δ of S to $M(S)$ by $\delta(s)f=f(s)(s\in S, f\in B(S))$. Then we see that $c^*(s)\delta(t)(f)=f(sts^*)=\delta(sts^*)f$ for any $s, t\in S$ and $f\in B(S)$. So $\{M(S), B(S)^*, c^*, \delta\}$ is a compact affine conjugate action of S. We denote by $\mathrm{Co}(A)$ the convex hull of any subset A of a linear space.

Theorem 5. *The following conditions are equivalent.*

(1) *S is inner amenable on V.*

(2) *For any given compact affine conjugate action $\{Q, T, \pi, \tau\}$ of S, there exists a point p in the closure of $\mathrm{Co}(\{\tau(s) ; s\in V\})$ such that $\pi(s)p=p$ for all $s\in S$.*

Sketch of proof. Let $\phi\in IM(S, V)$. We note that ϕ is in the w^*-closure of $\mathrm{Co}(\{\delta(s) ; s\in V\})$. According to Lemma 2.1 in [12], any compact affine conjugate action $\{Q, T, \pi, \tau\}$ of S induces a continuous affine map $\tau^\wedge$ of $M(S)$ to Q such that $\tau^\wedge(\delta(s))=\tau(s)$ for any $s\in S$ and $\tau^\wedge(c^*(s)\phi)=\pi(s)\tau^\wedge(\phi)$ for any $(s, \phi)\in S\times M(S)$. So $p=\tau^\wedge(\phi)$ is the desired point in the closure of $\mathrm{Co}(\{\tau(s) ; s\in V\})$. Conversely suppose (2). Applying (2) to the compact affine conjugate action $\{M(S), B(S)^*, c^*, \delta\}$ of S, we get an inner invariant mean in $M(S, V)$.

§3. Inner amenability on $S-Z_S$. As noted in Corollary 1, S is inner amenable on Z_S. In this section let us show some conditions for S to be inner amenable on $S-Z_S$. For brevity we write $G=G_S$ and $Z=Z(G_S)$. By virtue of Theorem 1, it suffices to consider the conditions for G to be inner amenable on $G-Z$ instead of the inner amenability of S on $S-Z_S$. Let $H(G)$ be the C^*-algebra of all bounded linear operators on the Hilbert space $L_2(G)$ and $C^*(c_2, G)$ be the C^*-subalgebra of $H(G)$ generated by the unitary operators $c_2(g)$, $g\in G$, on $L_2(G)$, where $c_2(g)h(a)=h(g^{-1}ag)$, $a\in G$, $h\in L_2(G)$. We define $P_Z\in H(G)$ by $P_Zh=1_Zh$, $h\in L_2(G)$. Let us fix a point w in Z and put $w=1_{\{w\}}\in L_2(G)$. The next theorem is a generalization of the main result in Paschke [8].

Theorem 6. *The following conditions are mutually equivalent.*

(1) *S is inner amenable on $S-Z_S$, i.e., G is inner amenable on $G-Z$.*

(2) *P_Z is not contained in $C^*(c_2, G)$.*

(3) *There exists a state ω on $H(G)$ such that $\omega(P_Z)=0$ and $\omega(c_2(g))=1$ for all $g\in G$.*

Sketch of proof. Suppose (1). Since G is inner amenable both on $G-Z$ and Z, it follows from Proposition 4.7 in [6] that $\|P_Z-T\|\geqq 1/2$ for any $T\in C^*(c_2, G)$. So (1) implies (2). We note that $P_Zc_2(g)=c_2(g)P_Z=P_Z$ for any $g\in G$ and $P_ZT=TP_Z=\langle Tw, w\rangle P_Z$ for any $T\in C^*(c_2, G)$. So it follows from (2) that the direct sum $C^*(c_2, P_Z, G)$ of $C^*(c_2, G)$ and the 1-dimensional algebra generated by P_Z becomes also a C^*-subalgebra of $H(G)$. Let us define a state ϕ on $C^*(c_2, P_Z, G)$ by $\phi(T+cP_Z)=\langle Tw, w\rangle$, $T\in C^*(c_2, G)$, $c\in\mathbf{C}$. Then $\phi(P_Z)=0$ and $\phi(c_2(g))=1$ for any $g\in G$. Hence any extending state ω on $H(G)$ of ϕ satisfies (3). Finally let ω be a state on $H(G)$ as in (3). Then by the unitarity of $c_2(g)$, $\omega(c_2(g)T)=\omega(Tc_2(g))=\omega(T)$ for any $g\in G$ and $T\in H(G)$. For any $f\in B(G)$ we define $m(f)\in H(G)$ by $m(f)h=fh$, $h\in L_2(G)$, and define $\phi\in M(G)$ by

$\phi(f) = \omega(m(f))$, $f \in B(G)$. Since $m(c(g)f) = c_2(g)^{-1}m(f)c_2(g)$, we have $\phi(c(g)f) = \omega(c_2(g)^{-1}m(f)c_2(g)) = \omega(f)$ for any $g \in G$ and $f \in B(G)$, and $\phi(Z) = \omega(m(1_Z)) = \omega(P_Z) = 0$. Therefore $\phi \in IM(S, V)$.

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