24. *An Example of Elliptic Curve over Q with Rank > 21*

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Abstract: We construct an elliptic curve over Q with rank ≥ 21.

In continuation of our previous paper [1], we give an example of elliptic curve over Q with Q-rank ≥ 21 using Mestre's method. As was explained in [1], any 6-ple $A = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \in Z^6$ gives rise to a $Q(T)$-polynomial $r_A(X) = \sum_{i=0}^{5} c_i X^i$, $c_i \in Q(T)$, where $c_5 = 0$ for suitable choices of A. In the following, A will be always chosen so that $c_5 = 0$. Then $Y^2 = r_A(X)$ gives an elliptic curve over $Q(T)$ which will be denoted by $\mathscr{E}_A$. Let $t \in Q$. The elliptic curve over Q obtained from $\mathscr{E}_A$ by specialization $T \to t$ is denoted by $E_{A,t}$. We have defined in [1] the number $S(N, E)$, $S'(N, E)$ for an integer N and an elliptic curve E over Q, and indicated that it is experimentally known that the Q-rank of E is found high when $S(N, E)$, $S'(N, E)$ are large.

Now let $A = (399,380,352,47,4,0)$. (Then we have $c_5 = 0$.) For $\mathscr{E}_A$, $E_{A,t}$ we shall write simply $\mathscr{E}$, E_t. We search in the family of curves

$$\{E_{t_1/t_2} \mid 1 \leq t_1 \leq 20000,\ 1 \leq t_2 \leq 2000,\ t_1\, t_2 \text{ are co-prime}\},$$

curves satisfying

$S(401) \geq 31.5$, $S'(401) \geq 11.5$, $S(1987) \geq 61$, $S'(1987) \geq 16.5$,
$S(3001) \geq 71$, $S'(3001) \geq 16.5$, $S(4003) \geq 75$, $S'(4003) \geq 16.5$,
$S(5297) \geq 80$, $S'(5297) \geq 17.5$, $S(6581) \geq 84$, and $S'(6581) \geq 20$,

and find $E_{1393/216}$, $E_{1649/12}$, $E_{6629/348}$, $E_{8057/876}$, and $E_{14721/376}$, for the last of which we could show that the Q-rank ≥ 21. Thus we have

Theorem. *Q-rank of $E_{14721/376}$ is ≥ 21.*

In fact, $E_{14721/376}$ is Q-isomorphic to the minimal Weiestrass model

$$y^2 + xy + y = x^3 + x^2 - 215843772422443922015169952702159835x$$
$$- 19474361277787151947255961435459054151501792241320535$$

whose conductor is $2*3*5*7*13*17*23*47*4507*$
$11548261137426760214116839824139660869938190231961$
$\backslash 72257362976160612359767 19.$

On this curve the following $P_1, \ldots, P_{21}$ are independent points.

$P_1 = [8008430088893400659 33/16, 226622141909109039907835847653 47/64]$

$P_2 = [1061054106676391459063 7/2209,$
$\quad 10877441148251784548400947947780 34/103823]$

$P_3 = [907186946780634143, 7289163861684518306416776 98]$

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$P_4 = [19683320108556444421940831 07/227919409,$
$227780739893044081958741018479392376 3894/3440899317673]$
$P_5 = [18546347413906465252800007 5/366301321,$
$225699857838583242849473830466481978146/7010640982619]$
$P_6 = [-1248526107123469143 2503/123904,$
$154330335342893998228217175 2702539/43614208]$
$P_7 = [-597030140876847470 37/361, 74188124509415406852503612 6962/6859]$
$P_8 = [-732704634047996130 67/361, 86687813785863879289111794 3482/6859]$
$P_9 = [-3607333963986275 65, 1069858404840967289478839 74]$
$P_{10} = [-3894451809579068 97, 7428835511879067385254209 8]$
$P_{11} = [-1474458350349858512665407/14205361,$
$227849340157836808431040902825933263 2/53540005609]$
$P_{12} = [-11430585603546889269177927 7/278589481,$
$1697277976887713629284102963998709537 8/4649937027371]$
$P_{13} = [-219725336008282027 97/81, 1007907865849635045638760053 02/729]$
$P_{14} = [-25047938415396324842058977/71216721,$
$683471925669849430075220526129377520 62/600997908519]$
$P_{15} = [343482808188511835221371528 4707/5137262501809,$
$42799124838389250442349391653296975768124338 46/11643877735262694377]$
$P_{16} = [-2276563132616766 47, 1336600243272689490952977 98]$
$P_{17} = [-40980894341059921378352 93/12552849,$
$566008841399135175930140365989088970 6/44474744007]$
$P_{18} = [265782873586917802021261 7/1495729,$
$417449973154999718659613172127320137 6/1829276567]$
$P_{19} = [883965004314243424124994323/850947241,$
$232500779860022149171450417087212768121 78/24822981967211]$
$P_{20} = [37543938954172817209003/73441,$
$122409791599128009990383549002029 8/19902511]$
$P_{21} = [191653123475024584101622 33/17214201,$
$755938398157414854503489970555516949 52/71421719949]$

By using calculation system PARI, we have that the determinant of the matrix $(\langle P_i, P_j \rangle)_{1 \leq i,j \leq 21}$ assosiated to canonical height is 10576626830616579980798 87.489. Since this determinant is non-zero, we see that $P_1, \ldots, P_{21}$ are independent points.

Remark 1. The Q-rank of the elliptic curve $E_{A,t}$ with the values of A, t being $(346,260,255,146,55,0)$, $5081/94$ respectively has been shown at least 20.

Remark 2. We used Hitachi super computer Hitac S-820 in Nihon University by the coutesies of Prof. Hideo Nagasaka in Nihon University and of Dr. Makoto Murofushi in Polytechnic University to whom the authors would like to express their thanks.

Reference

[1] K. Nagao: An example of elliptic curve over Q with rank ≥ 20. Proc. Japan Acad., **69A**, 291–293 (1993).