31. *On the Second Microlocalization along Isotropic Submanifolds*

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1. Introduction. Around 1985, Lebeau [9] developed the theory of the second microlocalization along isotropic submanifolds, and defined the "Γ-analytic microfunction" that had the unique continuation properties along bicharacteristic leaves. In this paper, we will give an explicit representation using the boundary values of holomorphic functions as Okada-Tose [10] did in the case of regular involutive submanifolds. As a consequence, we prove that "Γ-analytic microfunctions" form a strictly wider subclass of the microfunctions than that of the microfunctions with holomorphic parameters. Moreover, we can show the unique continuation properties of "Γ-analytic microfunctions" elementarily using the local version of Bochner's tube theorem [8]. Note that our methods of proofs are completely different from Lebeau's. We shall announce the results which will be proved in the subsequent paper [7].

2. Lebeau's second FBI-transformation. Let M be open in $\boldsymbol{R}^n_x = \boldsymbol{R}^d_{x'} \times \boldsymbol{R}^{n-d}_{x''}$, X be its complexification. We take coordinates of $T^*_M X \simeq \sqrt{-1}\, T^*M$ as $(x', x''; \sqrt{-1}\,\xi' dx' + \sqrt{-1}\,\xi'' dx'')$, and define its regular involutive submanifold Λ as $\{\xi' = 0\}$. Let $\dot{\xi}'' \in \boldsymbol{R}^{n-d}$ be a fixed non zero vector, and set

(2.1) $$\Gamma := \{x'' = 0,\ \xi' = 0,\ \xi'' = \dot{\xi}''\} \subset \Lambda \subset \sqrt{-1}\,T^*M.$$

Note that Γ is a bicharacteristic leaf of Λ. The following imbedding is intrinsically defined by Lebeau [9]

(2.2) $$\dot{T}^*\Gamma \ni (x'; \xi^{*\prime} dx') \mapsto (x' - \sqrt{-1}\,\xi^{*\prime}, 0) \in \boldsymbol{C}^d_{z'} \times \boldsymbol{C}^{n-d}_{z''} = \boldsymbol{C}^n_z = X.$$

Let $u(x)$ be a hyperfunction with compact support. We follow some definitions of Lebeau [9].

Definition 2.1 (Second FBI-transformation). We define the second FBI-transformation of u along Γ by

(2.3) $$T^2_\Gamma u(z; \lambda, \mu) := \int_{\boldsymbol{R}^n} u(x) \exp\Big[-\frac{\lambda\mu^2}{2}(z' - x')^2 - \frac{\lambda}{2}(\mu z'' - x'')^2 - \sqrt{-1}\,\lambda x''\cdot\dot{\xi}''\Big] dx,$$

where $z = (z', z'') \in \boldsymbol{C}^d_{z'} \times \boldsymbol{C}^{n-d}_{z''}$ and $\lambda, \mu > 0$ are parameters.

Lebeau [9] has defined the second wave front set $S^2_\Gamma u$ of u as a closed subset of $\boldsymbol{C}^n$ in terms of $T^2_\Gamma u$ as follows.

Definition 2.2. For $z \in \boldsymbol{C}^n$, $z \notin S^2_\Gamma u$, if $\exists\, U$: a neighborhood of z in $\boldsymbol{C}^n$, $\exists\, \mu_0 > 0$, $\exists\, \delta > 0$, and

(2.4) $\exists\, f : (0, \mu_0) \to \boldsymbol{R}_+$ (a decreasing function) such that, (2.5) $$| T_\Gamma^2 u(z\,;\lambda,\mu) | \leq \exp\Big[\frac{\lambda\mu^2}{2}|\mathrm{Im}\, z|^2 - \delta\lambda\mu^2\Big]$$

for $\forall z \in U$, $0 < \forall \mu < \mu_0$, and $f(\mu) < \forall \lambda$.

Considering $\dot{T}^*\Gamma$ as a subset of $\boldsymbol{C}^n$ by (2.2), we set

(2.6) $$\dot{S}S_\Gamma^2 u := \dot{T}^*\Gamma \cap S_\Gamma^2 u.$$

This is a closed $\boldsymbol{R}_+$-conic subset of $\dot{T}^*\Gamma$. Though $\dot{S}S_\Gamma^2 u$ is defined for hyperfunctions with compact supports, this definition is also valid for $u \in \mathscr{C}_M|_\Gamma$ ($\mathscr{C}_M$ is a sheaf of the microfunctions) as the case of the usual analytic wave front set of Sjöstrand [12]. This assures the following definition.

Definition 2.3 (Γ-analytic microfunctions). For $u \in \mathscr{C}_M|_\Gamma$, u is a Γ-analytic microfunction, if $\dot{S}S_\Gamma^2 u = \emptyset$.

Then Γ-analytic microfunctions form a subsheaf of $\mathscr{C}_M|_\Gamma$.

3. Characterization by defining holomorphic functions. In this section, we give an equivalent condition of the Γ-analyticity in terms of the boundary values of holomorphic functions.

Theorem 3.1 (An expression by the boundary values of holomorphic functions). *Following two conditions are equivalent for* $\forall u \in \mathscr{C}_M|_\Gamma$.

(i) *u is a Γ-analytic microfunction at $x' = 0$.*

(ii) *u can be expressed as a section of $\mathscr{C}_M|_\Gamma$ in the following way:*

(3.1) $$u(x) = \Big[\int_{|\xi''-\dot{\xi}''|\leq\varepsilon} F(x+\sqrt{-1}(0,\dot{\xi}'')0,\xi'')\,d\sigma(\xi'')\Big],$$

where the integration is taken on the unit sphere of $\boldsymbol{R}^{n-d}_{\xi''}$, $F(z,\xi'')$ is a C^∞-function, holomorphic in z-variables, defined on

(3.2) $$\Omega := \{(z', z'', \xi'') \in \boldsymbol{C}^d \times \boldsymbol{C}^{n-d} \times \boldsymbol{R}^{n-d}\,;\, |z| < \varepsilon,\ |\xi''| = 1,$$
$$|\xi'' - \dot{\xi}''| \leq \varepsilon,\ \mathrm{Im}\, z''\cdot\xi'' > C\cdot(|\mathrm{Re}\, z''|^2 + |\xi'' - \dot{\xi}''|^2)\cdot|\mathrm{Im}\, z'|\}$$

for some $\varepsilon > 0$, $C > 0$.

The above theorem is obtained by the inversion formula for the second FBI-transformation T_Γ^2 in Takeuchi [13].

Corollary 3.2. *The Γ-analytic microfunctions form a strictly wider subclass of $\mathscr{C}_M|_\Gamma$ than that of the microfunctions with holomorphic parameters in z'.*

Example 3.3. If $n - d = 1$, Γ is considered as a Lagrangean submanifold in $\sqrt{-1}S^*M$. In this case, we may take $\dot{\xi}'' = 1$, and then condition (ii) of the above theorem reduces to

ii) $(n-d=1)$ $u(x) = [F(x+\sqrt{-1}(0,\dot{\xi}'')0)]$, where F is holomorphic on

(3.3) $$\Omega : = \{(z', z'') \in \boldsymbol{C}^d \times \boldsymbol{C}^1\,;\, |z| < \varepsilon,\ \mathrm{Im}\, z'' > C\cdot(|\mathrm{Re}\, z''|^2 \times |\mathrm{Im}\, z'|\}$$

for some $\varepsilon > 0$, $C > 0$.

As a consequence of Okada-Tose [10], if $u \in \mathscr{C}_{x''}\mathscr{O}_{z'}$ (microfunctions with holomorphic parameters in z'-variables), we can take a defining holomorphic function F of u such that F is holomorphic on

(3.4) $$\Omega' := \{(z', z'') \in \boldsymbol{C}^d \times \boldsymbol{C}^1\,;\, |z| < \varepsilon,\ \mathrm{Im}\, z'' > 0\}.$$

It is easy to see that Ω is essentially smaller than Ω', which implies Corollary 3.2.

4. Unique continuation properties of Γ-analytic microfunctions. The following theorems were essentially proved by Lebeau [9] by means of Hadamard's three circles theorem. Here we prove them in a completely different way, with the help of the explicit expressions of the Γ-analytic microfunctions in Theorem 3.1.

Theorem 4.1 (Unique continuation properties). *If $u \in \mathscr{C}_M|_\Gamma$ is a Γ-analytic microfunction, then u has the unique continuation properties along Γ.*

If $n - d = 1$, the proof of this theorem reduces to the local version of Bochner's tube theorem for the domain of type (3.3). For the general case, shrinking the domain of integration (3.1), we sweep out the microanalyticities along Γ by using the local version of Bochner's tube theorem.

Remark 4.2. From the proof, we can find non Γ-analytic microfunctions which still have the unique continuation properties along Γ. Indeed, we will get a stronger result concerning the unique continuation properties only by using the local version of Bochner's tube theorem.

These methods are also valid for proving the microlocal Holmgren type theorem below.

Theorem 4.3. *Let p be a point of Γ, ϕ be a real-valued real analytic function on Γ with $\phi(p) = 0$, $d\phi(p) \neq 0$, and we assume for $u \in \mathscr{C}_M|_\Gamma$,*

$$\pm\, d\phi(p) \notin \dot{S}S^2_\Gamma u. \tag{4.1}$$

Then, for a neighborhood U of p,

$$u \equiv 0 \text{ in } U \cap \{\phi < 0\} \Rightarrow u \equiv 0 \text{ in a neighborhood of } p. \tag{4.2}$$

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