76. *Criteria for the Finiteness of Restriction of $U(\mathfrak{g})$-modules to Subalgebras and Applications to Harish-Chandra Modules*

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(Communicated by Kiyosi ITÔ, M. J. A., Dec. 14, 1992)

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, and $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. In this paper, we give simple and useful criteria for finitely generated $U(\mathfrak{g})$-modules H to remain finite under the restriction to subalgebras $A \subset U(\mathfrak{g})$, by using the algebraic varieties in $\mathfrak{g}^*$ associated to H and A. It is shown that, besides the finiteness, the $U(\mathfrak{g})$-modules H satisfying our criteria preserve some important invariants under the restriction.

Applying the criteria to Harish-Chandra modules of a semisimple Lie algebra $\mathfrak{g}$, we specify among other things, a large class of Lie subalgebras of $\mathfrak{g}$ on which all the Harish-Chandra modules are of finite type. This allows us to extend largely the finite multiplicity theorems for induced representations of a semisimple Lie group, established in our earlier work [8].

1. Associated varieties for finitely generated $U(\mathfrak{g})$-modules. We begin with defining three important invariants: the associated variety, the Bernstein degree and the Gelfand-Kirillov dimension, of finitely generated modules over a complex Lie algebra (cf. [6]).

Let V be a finite-dimensional complex vector space. We denote by $S(V) = \bigoplus_{k=0}^{\infty} S^k(V)$ the symmetric algebra of V, where $S^k(V)$ is the homogeneous component of $S(V)$ of degree k. Let $M = \bigoplus_{k=0}^{\infty} M_k$ be a finitely generated, nonzero, graded $S(V)$-module, on which $S(V)$ acts in such a way as $S^k(V)\, M_{k'} \subset M_{k+k'}$ $(k, k' \geq 0)$. Then each homogeneous component M_k of M is finite-dimensional.

Proposition 1 (Hilbert-Serre, see [9, Ch. VII, §12]). (1) *There exists a unique polynomial $\varphi_M(q)$ in q such that $\varphi_M(q) = \dim(M_0 + M_1 + \cdots + M_q)$ for sufficiently large q.*

(2) *Let $(c(M)/d(M)!)q^{d(M)}$ be the leading term of φ_M. Then $c(M)$ is a positive integer, and the degree $d(M)$ of this polynomial coincides with the dimension of the associated algebraic cone*

(1.1) $\nu(M) := \{\lambda \in V^* \mid f(\lambda) = 0 \text{ for all } f \in \mathrm{Ann}_{S(V)}M\}.$

Here, $\mathrm{Ann}_{S(V)}M$ denotes the annihilator of M in $S(V)$, V^* the dual space of V, and $S(V)$ is identified with the polynomial ring over V^* in the canonical way.

For a finite-demensional complex Lie algebra $\mathfrak{g}$, let $(U_k(\mathfrak{g}))_{k=0,1\ldots}$ denote the natural filtration of enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, where $U_k(\mathfrak{g})$ is the subspace of $U(\mathfrak{g})$ generated by elements $X_1 \ldots X_m$ with $m \leq k$ and $X_j \in \mathfrak{g}$ $(1 \leq j \leq m)$. We identify the associated commutative ring $\mathrm{gr}\, U(\mathfrak{g}) = \bigoplus_{k \geq 0}$

$U_k(\mathfrak{g})/U_{k-1}(\mathfrak{g})$ ($U_{-1}(\mathfrak{g}) := (0)$) with the symmetric algebra $S(\mathfrak{g}) = \bigoplus_{k\geq 0} S^k(\mathfrak{g})$ of $\mathfrak{g}$ in the canonical way.

Now let H be a finitely generated, non-zero $U(\mathfrak{g})$-module. Take a finite-dimensional generating subspace H_0 of $H : H = U(\mathfrak{g})H_0$. Setting $H_k = U_k(\mathfrak{g})H_0$ for $k = 1,2,\dots$, and $H_{-1} = (0)$, we get a finitely generated, graded $S(\mathfrak{g})$-module

(1.2) $$M = \mathrm{gr}(H\,;\,H_0) := \bigoplus_k M_k \text{ with } M_k = H_k/H_{k-1}.$$

The variety $\nu(M) \subset \mathfrak{g}^*$, the integers $c(M)$ and $d(M)$ defined for this M as in Proposition 1, are independent of the choice of a generating subspace H_0. These three invariants of H are called respectively the *associated variety*, the *Bernstein degree* and the *Gelfand-Kirillov dimension* of H. We denote $\nu(M)$, $c(M)$ and $d(M)$ respectively by $\nu(\mathfrak{g}\,;H)$, $c(\mathfrak{g}\,;H)$ and $d(\mathfrak{g}\,;H)$, emphasizing that H is being considered as a $U(\mathfrak{g})$-module.

2. Restriction of $U(\mathfrak{g})$-modules to subalgebras. Let A be a subalgebra of $U(\mathfrak{g})$ containing the identity element $1 \in U(\mathfrak{g})$. We denote by R the graded subalgebra of $S(\mathfrak{g}) = \mathrm{gr}\, U(\mathfrak{g})$ associated to A: $R = \mathrm{gr}\, A := \bigoplus_{k\geq 0} A_k/A_{k-1}$ with $A_k = A \cap U_k(\mathfrak{g})$. We say that a finitely generated $U(\mathfrak{g})$-module H has the *good restriction* to A if there exists a finite-dimensional generating subspace H_0 of H for which the associated graded $S(\mathfrak{g})$-module $\mathrm{gr}(H\,;\,H_0)$ is finitely generated over R.

The following theorem characterizes, by means of the associated varieties, the $U(\mathfrak{g})$-modules H having the good restriction to A.

Theorem 1. (1) *The restriction of H to A is good whenever the condition*

(2.1) $$\nu(\mathfrak{g}\,;H) \cap R_+^\# = (0)$$

on algebraic varieties in $\mathfrak{g}^*$ is satisfied. Here $R_+^\# := \{\lambda \in \mathfrak{g}^* \mid f(\lambda) = 0$ for all $f \in R_+\}$ denotes the variety in $\mathfrak{g}^*$ associated to the maximal graded ideal $R_+ := \bigoplus_{k>0}(R \cap S^k(\mathfrak{g}))$ of $R = \mathrm{gr}\, A$.

(2) *Conversely, if R is Noetherian and if $H \neq (0)$ admits the good restriction to A, one necessarily has* (2.1).

Let $\mathfrak{n}$ be a Lie subalgebra of $\mathfrak{g}$. Applying this theorem to the case $A = U(\mathfrak{n})$ ($R = S(\mathfrak{n})$ is obviously Noetherian), we obtain immédiately the following

Corollary 1. *A finitely generated $U(\mathfrak{g})$-module $H \neq (0)$ has the good restriction to $U(\mathfrak{n})$ if and only if $\nu(\mathfrak{g}\,;H) \cap \mathfrak{n}^\perp = (0)$ holds, where $\mathfrak{n}^\perp := \{\lambda \in \mathfrak{g}^* \mid \langle\lambda, X\rangle = 0$ for all $X \in \mathfrak{n}\}$ is the orthogonal of $\mathfrak{n}$ in $\mathfrak{g}^*$.*

The $U(\mathfrak{g})$-modules admitting the good restriction enjoy nice properties as follows.

Theorem 2. *Suppose that H has the good restriction to a subalgebra $A \subset U(\mathfrak{g})$.*

(1) *H is finitely generated as an A-module.*

(2) *If $A = U(\mathfrak{n})$ for a Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$, then H is of finite type over $U(\mathfrak{n})$, and so one can define the associated variety $\nu(\mathfrak{n}\,;H)$, Bernstein degree $c(\mathfrak{n}\,;H)$, and Gelfand-Kirillov dimension $d(\mathfrak{n}\,;H)$ of H as a $U(\mathfrak{n})$-module as well as those as a $U(\mathfrak{g})$-module. These two kinds of quantities have the relations*

(2.2) $$c(\mathfrak{g}\,;H)=c(\mathfrak{n}\,;H),\ d(\mathfrak{g}\,;H)=d(\mathfrak{n}\,;H),$$

and hence

(2.3) $$\dim \nu(\mathfrak{g}\,;H)=\dim \nu(\mathfrak{n}\,;H).$$

Moreover one has

(2.4) $$p^*\nu(\mathfrak{g}\,;H)\subset \nu(\mathfrak{n}\,;H),$$

where $p^*:\mathfrak{g}^*\to\mathfrak{n}^*$ *denotes the restriction map of linear forms.*

We can give two interesting consequences of the above theorems, as follows.

Corollary 2. *Let* $\mathfrak{n}$ *be a Lie subalgebra of* $\mathfrak{g}$, *and* H *be a finitely generated* $U(\mathfrak{g})$*-module satisfying the condition* $\nu(\mathfrak{g}\,;H)\cap\mathfrak{n}^{\perp}=(0)$. *Then, the* $\mathfrak{n}$*-homology groups* $H_k(\mathfrak{n},H)$ $(k=0,1,\dots)$ *of* H *(see e.g.,* [2] *for the definition) are all finite-demensional.*

Corollary 3. *If a finitely generated* $U(\mathfrak{g})$*-module* H *has the good restriction to* $U(\mathfrak{n})$, *the Gelfand-Kirillov dimension* $d(\mathfrak{g}\,;H)$ *of* H *does not exceed* $\dim \mathfrak{n}$.

3. Nilpotent variety $\mathcal{N}(\mathfrak{p})$ and good restriction of Harish-Chandra modules. Now, assume $\mathfrak{g}$ to be semisimple, and let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be a symmetric decomposition of $\mathfrak{g}$ determined by an involutive automorphism of $\mathfrak{g}$. We consider the category $C(\mathfrak{k})$ of finitely generated $U(\mathfrak{g})$-modules H on which the subalgebra $U(\mathfrak{k})Z(\mathfrak{g})$ acts locally finitely, where $Z(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$. Such an H in $C(\mathfrak{k})$ is called a *Harish-Chandra* $(\mathfrak{g},\mathfrak{k})$*-module.* We regard the varieties $\nu(\mathfrak{g}\,;H)\subset\mathfrak{g}^*$ as algebraic cones in $\mathfrak{g}$ by identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form of $\mathfrak{g}$.

Lemma. *The associated variety* $\nu(\mathfrak{g}\,;H)$ *of any Harish-Chandra* $(\mathfrak{g},\mathfrak{k})$*-module* H *is contained in the variety* $\mathcal{N}(\mathfrak{p})$ *of all nilpotent elements of* $\mathfrak{p}$. *Moreover, there exists an* $\tilde{H}$ *in* $C(\mathfrak{k})$ *such that* $\nu(\mathfrak{g}\,;\tilde{H})$ *coincides with the whole* $\mathcal{N}(\mathfrak{p})$.

Theorem 1 together with this lemma yields the following result.

Theorem 3. *All the Harish-Chandra* $(\mathfrak{g},\mathfrak{k})$*-modules have the good restriction to a subalgebra* A *of* $U(\mathfrak{g})$ *if* $\mathcal{N}(\mathfrak{p})\cap R_+^{\#}=(0)$ *holds for* $R=\mathrm{gr}\,A$. *The converse is also true provided that* R *is Noetherian.*

4. Large Lie subalgebras of a real semisimple Lie algebra. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, and $\mathfrak{g}_0=\mathfrak{k}_0\oplus\mathfrak{p}_0$ be the Cartan decomposition of $\mathfrak{g}_0$ determined by an involution θ. Conventionally, we write $\mathfrak{h}$ $(\subset\mathfrak{g})$ for the complexification of a real vector subspace $\mathfrak{h}_0$ of $\mathfrak{g}_0$ by dropping the subscript '0'.

A Lie subalgebra $\mathfrak{n}_0$ of $\mathfrak{g}_0$ is said to be *large* in $\mathfrak{g}_0$ if there exists an element $x\in\mathrm{Int}(\mathfrak{g}_0)$ for which every Harish-Chandra $(\mathfrak{g},\mathfrak{k})$-module has the good restiction to $U(x\cdot\mathfrak{n})$. By Theorem 3, this amounts to a simple geometric condition:

(4.1) $$(x\cdot\mathfrak{n})^{\perp}\cap\mathcal{N}(\mathfrak{p})=(0)\ \text{for some}\ x\in\mathrm{Int}(\mathfrak{g}_0).$$

Here $\mathrm{Int}(\mathfrak{g}_0)$ denotes the group of inner automorphisms of $\mathfrak{g}_0$. Notice that the largeness of a Lie subalgebra does not depend on the choice of a $\mathfrak{k}_0$.

We now specify many of large Lie subalgebras of $\mathfrak{g}_0$.

At first, here are two kinds of typical large Lie subalgebras.

Proposition 2. (1) *Let* $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_{p,0} + \mathfrak{u}_{m,0}$ *be an Iwasawa decomposition of* $\mathfrak{g}_0$. *Then the maximal nilpotent Lie subalgebra* $\mathfrak{u}_{m,0}$ *of* $\mathfrak{g}_0$ *is large.*

(2) *The symmetrizing Lie subalgebra* $\mathfrak{h}_0 = \{X \in \mathfrak{g}_0 \mid \sigma X = X\}$ *is large in* $\mathfrak{g}_0$ *for any involutive automorphism* σ *of* $\mathfrak{g}_0$.

The claim (1), together with Theorem 3, covers the results of Casselman-Osborne [3, Th.2.3] and Joseph [4, II, 5.6] on the restriction of Harish-Chandra modules to $\mathfrak{u}_m$. The second one allows us to deduce the finite multiplicity theorem of van den Ban, for the quasi-regular representation on $L^2(G/H)$, associated to a semisimple symmetric space G/H (cf. [8]).

Now let $\mathfrak{q}_0$ be any parabolic subalgebra of $\mathfrak{g}_0$, and $\mathfrak{q}_0 = \mathfrak{l}_0 + \mathfrak{u}_0$ with $\mathfrak{l}_0 = \mathfrak{q}_0 \cap \theta\mathfrak{q}_0$, be its Levi decomposition. Since the Levi component $\mathfrak{l}_0$ is reductive, one can define large Lie subalgebras of $\mathfrak{l}_0$ just in the same way.

The largeness of Lie subalgebras is preserved by parabolic induction.

Proposition 3. *If* $\mathfrak{h}_0$ *is a large Lie subalgebra of* $\mathfrak{l}_0$, *the semidirect product Lie subalgebra* $\mathfrak{h}_0 + \mathfrak{u}_0$ *is large in* $\mathfrak{g}_0$.

Let $\mathfrak{q}_{m,0} = \mathfrak{m}_0 + \mathfrak{a}_{p,0} + \mathfrak{u}_{m,0}$ be a minimal parabolic subalgebra of $\mathfrak{g}_0$, where $\mathfrak{m}_0$ denotes the centralizer of $\mathfrak{a}_{p,0}$ in $\mathfrak{k}_0$. We say that a Lie subalgebra $\mathfrak{n}_0$ of $\mathfrak{g}_0$ is *quasi-spherical* if there exists a $z \in \mathrm{Int}(\mathfrak{g}_0)$ such that $z \cdot \mathfrak{n}_0 + \mathfrak{q}_{m,0} = \mathfrak{g}_0$. This is equivalent to saying that, if G is a connected Lie group with Lie algebra $\mathfrak{g}_0$, the analytic subgroup of G corresponding to $\mathfrak{n}_0$ has an open orbit on the flag variety G/Q_m with Q_m a minimal parabolic subgroup of G (cf. [1], [5]).

It is easy to verify that the large Lie subalgebras in Proposition 2 are quasi-spherical. The next theorem is the principal result of this section.

Theorem 4. *Quasi-spherical Lie subalgebras are always large in* $\mathfrak{g}_0$.

Remark. One can see from Theorem 3, coupled with a recent result of Bien and Oshima, that the converse is also true in the above theorem under the assumption that a large Lie subalgebra $\mathfrak{n}_0$ is algebraic in $\mathfrak{g}_0$.

5. Finite multiplicity criteria for induced representations. Let G be a connected semisimple Lie group with finite center, and K be a maximal compact subgroup of G. The corresponding Lie algebras are denoted respectively by $\mathfrak{g}_0$ and $\mathfrak{k}_0$. We have a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ of $\mathfrak{g}_0$ as in §4.

Let H be a Harish-Chandra $(\mathfrak{g},\mathfrak{k})$-module on which the compact group K acts in such a way as

$$k \cdot v = \sum_{n=0}^{\infty} (1/n!) X^n v$$

for $v \in H$ and $k = \exp X$ with $X \in \mathfrak{k}_0$. Such an H is called a *Harish-Chandra* $(\mathfrak{g}, K)$*-module.* A fundamental theorem of Harish-Chandra says that the (irreducible) Harish-Chandra $(\mathfrak{g}, K)$-modules correspond to the (irreducible) admissible representations of G, by passing to the K-finite part (see e.g., [7, Chap.8]).

If (η, E) is a smooth Fréchet representation (cf. [8, I, 2.1]) of a closed subgroup N of G, the group G acts on the space $\mathscr{A}(G\,;\,\eta)$ of real analytic functions $f : G \to E$ satisfying

$$f(gn) = \eta(n)^{-1} f(g) \text{ for } (n, g) \in N \times G,$$

by left translation L. $\mathscr{A}(G\,;\eta)$ has the structure of a $U(\mathfrak{g})$-module through differentiation. We call the gained $(L, \mathscr{A}(G\,;\eta))$ the *G-representation* or the *$U(\mathfrak{g})$-module analytically induced from η.*

We study $U(\mathfrak{g})$-homomorphisms from a Harish-Chandra $(\mathfrak{g}, K)$-module H into $\mathscr{A}(G\,;\eta)$, and especially the intertwining numbers

(5.1) $$I_{U(\mathfrak{g})}(H, \mathscr{A}(G\,;\eta)) := \dim \mathrm{Hom}_{U(\mathfrak{g})}(H, \mathscr{A}(G\,;\eta)),$$

which give the multiplicities of H in $\mathscr{A}(G\,;\eta)$ as $U(\mathfrak{g})$-submodules for irreducible H's.

For a Harish-Chandra $(\mathfrak{g}, K)$-module H, we can and do take a finite-dimensional, K-stable generating subspace H_0 of H. Then the associated graded $S(\mathfrak{g})$-module $M = \mathrm{gr}(H\,;H_0) = \oplus_k M_k$ has a natural K-module structure.

The intertwining number $I_{U(\mathfrak{g})}(H, \mathscr{A}(G\,;\eta))$ from H to $\mathscr{A}(G\,;\eta)$ can be estimated as in

Proposition 4. *For each $x \in G$, one has the inequality*

(5.2) $$I_{U(\mathfrak{g})}(H, \mathscr{A}(G\,;\eta)) \leq \sum_{k=0}^{\infty} \dim \mathrm{Hom}_{K\cap xNx^{-1}}(M_k/((x\cdot\mathfrak{n})M)_k, E_x),$$

where $((x\cdot\mathfrak{n})M)_k = M_k \cap (x\cdot\mathfrak{n})M)$ with $x\cdot\mathfrak{n} = \mathrm{Ad}(x)\mathfrak{n}$, is $(K \cap xNx^{-1})$-stable, and (η_x, E_x) denotes the representation of xNx^{-1} on E defined by $\eta_x(xnx^{-1}) = \eta(n)$ $(n \in N)$.

This proposition together with Theorem 1, enables us to deduce a useful criterion for the finiteness of intertwining numbers $I_{U(\mathfrak{g})}(H, \mathscr{A}(G\,;\eta))$, as follows.

Theorem 5. *The intertwining number $I_{U(\mathfrak{g})}(H, \mathscr{A}(G\,;\eta))$ from a Harish-Chandra $(\mathfrak{g}, K)$-module H to an induced $U(\mathfrak{g})$-module $\mathscr{A}(G\,;\eta)$ takes finite value if there exists an $x \in G$ such that*

(5.3) $$\nu(\mathfrak{g}\,;H) \cap (x\cdot\mathfrak{n})^{\perp} = (0),$$

and that

(5.4) $$\dim \mathrm{Hom}_{K\cap xNx^{-1}}(V_\gamma, E_x) < \infty \text{ holds}$$

for every irreducible constituent V_γ of $(K \cap xNx^{-1})$-module $M/(x\cdot\mathfrak{n})M$. Here $M = \mathrm{gr}(H\,;H_0)$ with K-stable H_0, and $\nu(\mathfrak{g}\,;H)$ is the associated variety of H.

We say that the induced module $\mathscr{A}(G\,;\eta)$ has the *finite multiplicity property* if the intertwining number $I_{U(\mathfrak{g})}(H, \mathscr{A}(G\,;\eta))$ is finite for every Harish-Chandra $(\mathfrak{g}, K)$-module H. As a consequence of Theorem 5, we establish

Theorem 6. *Let N be a closed subgroup of G whose Lie algebra $\mathfrak{n}_0$ is large in $\mathfrak{g}_0$, and take an element $x \in G$ such that $(x\cdot\mathfrak{n})^{\perp} \cap \mathscr{N}(\mathfrak{p}) = (0)$. Then, for a smooth Fréchet representation (η, E) of N, the induced module $\mathscr{A}(G\,;\eta)$ has the finite multiplicity property if so is the restriction of η to the compact subgroup $x^{-1}Kx \cap N$.*

Corollary 4. *If $\mathfrak{n}_0 = \mathrm{Lie}(N)$ is large in $\mathfrak{g}_0$, the representation $(L, \mathscr{A}(G\,;\eta))$ is of multiplicity finite for any finite-dimensional N-representation η.*

The above theorem extends one of the principal results in our previous work [8, I, Th.2.12], where we studied the case of semidirect product large

Lie subalgebras $\mathfrak{n}_0 = \mathfrak{h}_0 + \mathfrak{u}_0$ specified in Proposition 3 with symmetrizing $\mathfrak{h}_0$, through the theory of (K, N)-spherical functions.

The details of this article will appear elsewhere.

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