57. *An Explicit Expression of the Harish-Chandra C-function of SU(n, 1) Associated to the Ad(K) Representation*

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§0. Introduction. Let G be a semisimple Lie group with finite center, K a maximal compact subgroup of G. Let θ be the Cartan involution of G fixing K. Let $G = KAN$ be an Iwasawa decomposition of G and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ the corresponding decomposition of the Lie algebra $\mathfrak{g}$ of G. An element g of G, can then be uniquely written as $g = \kappa(g)\exp H(g)n(g)$ ($\kappa(g) \in K$, $H(g) \in \mathfrak{a}$, $n(g) \in N$). Put $\bar{N} = \theta N$ and let M be the centralizer of A in K. Let τ be a finite dimensional unitary representation of K and denote its representation space by V. The following operator given by the integral

$$C_\tau(\lambda) = \int_{\bar{N}} \tau(\kappa(\bar{n}))\, e^{-(\lambda+\rho)(H(\bar{n}))}\, d\bar{n} \qquad (\lambda \in \mathfrak{a}_C^*)$$

is called *Harish-Chandra's C-function associated to* τ (see [3]). It is well known that the operator $C_\tau(\sigma\,;\lambda)$ obtained by restricting $C_\tau(\lambda)$ to an irreducible M-component $V_\sigma(\subset V)$, is closely related to the intertwining operator between induced representations (see [3], [4]), and also in some special cases it can be represented by a diagonal matrix having diagonal elements in the form of quotients of products of gamma factors with respect to a certain orthogonal basis (cf. [1], [7]). Though it has been believed for a long time that this phenomena would be true for more general cases, even the computation of the determinant of $C_\tau(\sigma\,;\lambda)$ has not been easy (cf. [1], [2], [6]). We give here a series of affirmative examples to this conjecture in $G = SU(n, 1)$, $K = S(U(n) \times U(1))$ and $\tau = \mathrm{Ad}$ case, where Ad is the complex adjoint representation of K.

§1. Notation and preliminaries. Let $n (n \geq 2)$ be an integer and

$$G = SU(n, 1) = \{A \in GL(n+1, \boldsymbol{C})\,;\, {}^t\bar{A}I_{n,1}A = I_{n,1} \text{ and } \det A = 1\},$$

where

$$I_{n,1} = \begin{pmatrix} I_n & \\ & -1 \end{pmatrix} \in GL(n+1, \boldsymbol{C})$$

and I_n is the unit matrix of order n. Let

$$\mathfrak{g} = \mathfrak{su}(n, 1) = \{X \in \mathfrak{gl}(n+1, \boldsymbol{C})\,;\, {}^t\bar{X}I_{n,1} + I_{n,1}X = 0 \text{ and } \mathrm{tr}X = 0\},$$

$$\mathfrak{k} = \{\begin{pmatrix} A & \\ & \sqrt{-1}t \end{pmatrix} : A \in \mathfrak{u}(n),\quad t \in \boldsymbol{R} \text{ and } -\mathrm{tr}A = \sqrt{-1}t\}.$$

Let

$$\mathfrak{a} = \{tH\,;\, t \in \boldsymbol{R}\}, \qquad H = \begin{pmatrix} & & & 1 \\ & & 0 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$$

$$\mathfrak{n} = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha} \text{ and } \bar{\mathfrak{n}} = \theta\mathfrak{n},$$

where α is the simple root of $(\mathfrak{g}, \mathfrak{a})$ which satisfies $\alpha(H) = 1$, $\mathfrak{g}_\beta$ is the root subspace of $\mathfrak{g}$ for each root β and θ is the Cartan involution of $\mathfrak{g}$ defined by $\theta X = I_{n,1} X I_{n,1} = -{}^t\bar{X}$ $(X \in \mathfrak{g})$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$. Let $K, A, N, \bar{N}$ denote the analytic subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ and $\bar{\mathfrak{n}}$, respectively. Then $G = KAN$ is the Iwasawa decomposition of G corresponding to the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ and we have

$$K = \left\{ \begin{pmatrix} A & \\ & (\det A)^{-1} \end{pmatrix} ; A \in U(n) \right\},$$

$$A = \left\{ \begin{pmatrix} \cosh t & & \sinh t \\ & I_{n-1} & \\ \sinh t & & \cosh t \end{pmatrix} ; t \in \boldsymbol{R} \right\},$$

$$\bar{N} = \left\{ P \begin{pmatrix} 1 & & & & \\ z_1 & 1 & & & \\ \vdots & & \ddots & & \\ z_{n-1} & & & 1 & \\ 1 - F & -\bar{z}_1 & \cdots & -\bar{z}_{n-1} & 1 \end{pmatrix} P^{-1} ; F = 1 + \frac{1}{2}\sum_{i=1}^{n-1} |z_i|^2 + iu, \quad u \in \boldsymbol{R},\ z_1, \cdots, z_{n-1} \in \boldsymbol{C} \right\},$$

where

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & 1 \\ & \sqrt{2} I_{n-1} & \\ -1 & & 1 \end{pmatrix}.$$

We can see that $\bar{N}$ can be identified with $\boldsymbol{C}^{n-1} \times \boldsymbol{R}$. For any

$$\bar{n}(z, u) = P \begin{pmatrix} 1 & & & & \\ z_1 & 1 & & & \\ z_{n-1} & & \ddots & 1 & \\ 1 - F & -\bar{z}_1 & \cdots & -\bar{z}_{n-1} & 1 \end{pmatrix} P^{-1} \in \bar{N}$$

let $\bar{n}(z, u) = \kappa(\bar{n}(z, u)) a(\bar{n}(z, u)) n(\bar{n}(z, u))$ be the Iwasawa decomposition of $\bar{n}(z, u)$. Then we can see that

$$a(\bar{n}(z, u)) = P \,\mathrm{diag}(|F|, 1, \ldots, 1, |F|^{-1}) P^{-1}, \tag{1.1}$$

$$\kappa(\bar{n}(z, u)) = \begin{pmatrix} (2 - F)/|F| & -\sqrt{2}\bar{z}_1/F & \cdots & -\sqrt{2}\bar{z}_{n-1}/F & 0 \\ \sqrt{2} z_1/|F| & 1 - |z_1|^2/F & \cdots & -z_1\bar{z}_{n-1}/F & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{2} z_{n-1}/|F| & -\bar{z}_1 z_{n-1}/F & \cdots & 1 - |z_{n-1}|^2/F & 0 \\ 0 & 0 & \cdots & 0 & F/|F| \end{pmatrix} \tag{1.2}$$

(cf. [5]). For a real vector space W we denote by $W_{\boldsymbol{C}}$ its complexification. Now consider the complex vector space $V = \mathfrak{k}_{\boldsymbol{C}}$ which is isomorphic to $\mathfrak{gl}(n, \boldsymbol{C})$. Let $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$ and $\mathfrak{z}$ be the center of $\mathfrak{k}$. Then we see that

$$\mathfrak{k}_{1\boldsymbol{C}} = \left\{ \begin{pmatrix} X & \\ & 0 \end{pmatrix} : X \in \mathfrak{sl}(n, \boldsymbol{C}) \right\},$$

$$\mathfrak{z}_{\boldsymbol{C}} = \boldsymbol{C}\begin{pmatrix} I_n & \\ & -n \end{pmatrix},$$

and $V = \mathfrak{k}_{1\boldsymbol{C}} \oplus \mathfrak{z}_{\boldsymbol{C}}$ is the $\mathrm{Ad}(K)$-irreducible decomposition. Since

$$M = \left\{ \begin{pmatrix} d & & \\ & X & \\ & & d \end{pmatrix} ; d^2 \det X = 1,\ X \in U(n-1) \right\},$$

$V = \bigoplus_{i=0}^{4} V_i$ is the $\mathrm{Ad}(M)$-irreducible decomposition, where

$$V_0 = \mathfrak{z}_{\boldsymbol{C}}$$

$$V_1 = \left\{ z \begin{pmatrix} 1-n & & \\ & I_{n-1} & \\ & & 0 \end{pmatrix} ; z \in \boldsymbol{C} \right\},$$

$$V_2 = \left\{ \begin{pmatrix} 0 & z_{12} & \cdots & z_{1n} & 0 \\ & & O & & \end{pmatrix} ; z_{12}, \cdots, z_{1n} \in \boldsymbol{C} \right\},$$

$$V_3 = \left\{ \begin{pmatrix} 0 & \\ z_{21} & \\ \vdots & O \\ z_{n1} & \\ 0 & \end{pmatrix} ; z_{21}, \cdots, z_{n1} \in \boldsymbol{C} \right\},$$

$$V_4 = \left\{ \begin{pmatrix} 0 & & \\ & X & \\ & & 0 \end{pmatrix} ; X \in \mathfrak{sl}(n-1, \boldsymbol{C}) \right\}.$$

We put

$$X_0 = \begin{pmatrix} I_n & \\ & -n \end{pmatrix},\ X_1 = \frac{1}{1-n}\begin{pmatrix} 1-n & & \\ & I_{n-1} & \\ & & 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 \\ & & O & & \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0 & \\ 1 & \\ 0 & O \\ \vdots & \\ 0 & \end{pmatrix},\quad X_4 = \begin{pmatrix} & 0 & 0 \\ & 1 & 0 \\ O & 0 & \cdot \\ & \vdots & \vdots \\ & 0 & 0 \end{pmatrix},\ (X_i \in V_i).$$

Let $\mathfrak{m}$ be the Lie algebra of M, $\mathfrak{m}_1 = [\mathfrak{m}, \mathfrak{m}]$ and put

$$\mathfrak{h} = \left\{ \begin{pmatrix} h_1 & & & \\ & \ddots & & \\ & & h_n & \\ & & & 0 \end{pmatrix} \in \mathfrak{gl}(n+1, \boldsymbol{C}) ; \sum_{j=1}^{n} h_j = 0 \right\},$$

$$\mathfrak{h}_{\mathrm{m}} = \left\{ \begin{pmatrix} 0 & & & & \\ & h_2 & & & \\ & & \ddots & & \\ & & & h_n & \\ & & & & 0 \end{pmatrix} \in \mathfrak{gl}(n+1, \boldsymbol{C}),\ \sum_{j=2}^{n} h_j = 0 \right\}.$$

Then $\mathfrak{h}$ and $\mathfrak{h}_{\mathrm{m}}$ are Cartan subalgebras of $\mathfrak{k}_{1\boldsymbol{C}}(\simeq \mathfrak{sl}(n, \boldsymbol{C}))$ and $\mathfrak{m}_{1\boldsymbol{C}}$, respectively. Let $\{\alpha_1, \cdots, \alpha_{n-1}\}$ be the fundamental root system of $(\mathfrak{k}_{1\boldsymbol{C}}, \mathfrak{h})$ defined by

$$\alpha_i(H) = h_i - h_{i+1},\ H = \begin{pmatrix} h_1 & & & \\ & \ddots & & \\ & & h_n & \\ & & & 0 \end{pmatrix},\ i = 1,2,\cdots,n-1.$$

Then $\{\alpha_2, \cdots, \alpha_{n-1}\}$ is the fundamental root system of $(\mathfrak{m}_{1\boldsymbol{C}}, \mathfrak{h}_{\mathrm{m}})$. For each i $(i = 0,1,2,3,4)$ X_i is the M-highest weight vector of V_i with respect to the lexicographic order defined by $\{\alpha_2, \cdots, \alpha_{n-1}\}$. In our case Harish-Chandra's C-function is given as follows:

$$C(\lambda) = \int_{\bar{N}} e^{-(\lambda+\rho)H(\bar{n})} \mathrm{Ad}\,\kappa(\bar{n})\, d\bar{n} \qquad (\lambda \in \mathfrak{a}_{\boldsymbol{C}}^*), \tag{1.3}$$

where ρ denotes the rho function and $d\bar{n}$ denotes a Haar measure on $\bar{N}$. Since $\lambda \in \mathfrak{a}_{\boldsymbol{C}}^*$ can be written in the form $\lambda = \mu_\lambda \alpha$ $(\mu_\lambda \in \boldsymbol{C})$ we identify λ with the complex number μ_λ. Thus ρ is identified with n and (1.1) implies

$$e^{-(\lambda+\rho)H(\bar{n})} = |F(z,u)|^{-\lambda-n}.$$

Therefore we get from (1.3)

$$C(\lambda) = c\int_{\boldsymbol{C}^{n-1}\times \boldsymbol{R}} |F(z, u)|^{-\lambda-n}\, \mathrm{Ad}\,\kappa(\bar{n}(z, u))\, dz d\bar{z} du, \tag{1.4}$$

where c is a certain constant coming from the normalization of measures.

§2. The main theorem. Paying attention to the fact that $C(\lambda)$ acts as a scalar operator on each M-irreducible subspace V_i, we denote by $c_i(\lambda)$ the scalar induced by the action of $C(\lambda)$ on V_i. Our result in this paper is the following

Theorem 2.1. *We have the following expressions;*

$$c_0(\lambda) = c\,\frac{(2\pi)^n \cdot 2^{-\lambda}\Gamma(\lambda)}{\Gamma\left(\frac{\lambda+n}{2}\right)^2}, \tag{2.1}$$

$$c_1(\lambda) = c\,\frac{(2\pi)^n \cdot 2^{-\lambda-2}\Gamma(\lambda)(\lambda-n)^2}{\Gamma\left(\frac{\lambda+n+2}{2}\right)^2}, \tag{2.2}$$

$$c_2(\lambda) = c_3(\lambda) = c\,\frac{(2\pi)^n \cdot 2^{-\lambda-1}\Gamma(\lambda)(\lambda-n+1)}{\Gamma\left(\frac{\lambda+n+3}{2}\right)\Gamma\left(\frac{\lambda+n-1}{2}\right)}, \tag{2.3}$$

$$c_4(\lambda) = c\,\frac{(2\pi)^n \cdot 2^{-\lambda}\Gamma(\lambda)(\lambda+n-2)}{\Gamma\left(\frac{\lambda+n+2}{2}\right)\Gamma\left(\frac{\lambda+n-2}{2}\right)(\lambda+n)}. \tag{2.4}$$

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