43. *Some Remarks on the Fifth Painlevé Equation on the Positive Real Axis*

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Consider an equation of the form

$$\text{(V)}\quad y''=\left(\frac{1}{2y}+\frac{1}{y-1}\right)y'^2-\frac{y'}{x}+\frac{\alpha}{x^2}(y-1)^2y+\frac{\gamma y}{x}-\frac{\delta y(y+1)}{y-1}$$

($'=d/dx$) on the positive real axis $x>0$, where α, γ and δ are real constants. This is a special case of the fifth Painlevé equation. If $\delta>0$, equation (V) admits a one-parameter family of solutions $\{Y(a,x);\ a\in\boldsymbol{R}\}$ satisfying $Y(a,x)\simeq ae^{-\sqrt{2\delta}\,x}x^{-\gamma/\sqrt{2\delta}-1}$ as $x\to+\infty$. Furthermore any real-valued solution $\varphi(x)$ satisfying $\varphi(x)\to 0$ as $x\to+\infty$ is written in the form $\varphi(x)=Y(a_0,x)$, where a_0 is some real constant (cf. [1]). In this note we show the existence of families of solutions with analogous properties near the regular singular point $x=0$.

1. We treat the following equations equivalent to (V).

Proposition ([1; Proposition 2.2]). *By* $y=\tanh^2 u$, *equation* (V) *is changed into*

$$\text{(E.0)}\qquad x(xu')'=\frac{\alpha}{2}\tanh u\cosh^{-2}u+\frac{\gamma}{4}x\sinh 2u+\frac{\delta}{8}x^2\sinh 4u$$

and, by $y=-\tan^2 u$, *equation* (V) *is changed into*

$$\text{(E.}-\text{)}\qquad x(xu')'=\frac{\alpha}{2}\tan u\cos^{-2}u+\frac{\gamma}{4}x\sin 2u+\frac{\delta}{8}x^2\sin 4u.$$

We obtain a one-parameter family of solutions near $x=0$.

Theorem 1. *Assume that* $\alpha>0$. *Then, for an arbitrary positive constant* C_0, *equation* (V) *admits a family of real-valued solutions* $\{Y_0(c,x);\ -C_0<c<C_0\}$ *satisfying*

$$Y_0(c,x)=cx^{\sqrt{2\alpha}}(1+O(x+|c|x^{\sqrt{2\alpha}})),$$
$$(d/dx)\,Y_0(c,x)=\sqrt{2\alpha}\,cx^{\sqrt{2\alpha}-1}(1+O(x+|c|x^{\sqrt{2\alpha}}))$$

on the interval $0<x<r_0$, *where* $r_0=r_0(C_0)$ *is a sufficiently small positive constant.*

Proof. Equation (E.0) is written in the form

$$(1)\qquad x(xu')'=u\left(\frac{\alpha}{2}+F_0(x,u)\right),$$

where $F_0(x,u)=O(x+u^2)$ for $|u|<1$, $0<x<1$. By $u=x^{\sqrt{\alpha/2}}w$ equation (1) is changed into

$$(2)\qquad x(xw')'+\sqrt{2\alpha}\,xw'=wF(x,w),$$

where

$$(3)\qquad F(x,w)=O(x+x^{\sqrt{2\alpha}}w^2)$$

for $|w|<x^{-\sqrt{\alpha/2}}$, $0<x<1$. Consider a system of integral equations of the form

$$(4)\quad \begin{cases} w(x) = \kappa + \dfrac{1}{\sqrt{2\alpha}} \displaystyle\int_0^x \frac{1}{\xi}\left(1 - \left(\frac{\xi}{x}\right)^{\sqrt{2\alpha}}\right) F(\xi, w(\xi))\, w(\xi)\, d\xi, \\ xw'(x) = \displaystyle\int_0^x \frac{1}{\xi}\left(\frac{\xi}{x}\right)^{\sqrt{2\alpha}} F(\xi, w(\xi))\, w(\xi)\, d\xi \end{cases}$$

with an arbitrary real constant κ, of which the solutions satisfy equation (2). Let K be an arbitrary positive constant. By the method of successive approximation, we can verify that, if $|\kappa| < K$, equation (4) possesses a solution $w(\kappa, x)$ satisfying

$$(5)\quad \begin{cases} w(\kappa, x) = \kappa(1 + O(x + \kappa^2 x^{\sqrt{2\alpha}})), \\ x(d/dx)\, w(\kappa, x) = \kappa\, O(x + \kappa^2 x^{\sqrt{2\alpha}})) \end{cases}$$

for $0 < x < x_0$, where $x_0 = x_0(K)$ is a sufficiently small positive constant. Putting $Y_0(c, x) = \tanh^2(x^{\sqrt{\alpha/2}}\, w(\kappa, x))$ and $c = \kappa^2$, we obtain a family of solutions $\{Y_0(c, x)\,;\, 0 \leq c < C_0\}$ of (V). In a similar way, using (E.$-$), we obtain the family for $-C_0 < c \leq 0$.

Theorem 2. *Assume that* $\alpha > 0$. *If* $\sigma(x)$ *is a real-valued solution of equation* (V) *satisfying* $\sigma(x) \to 0$ *as* $x \to +0$, *then* $\sigma(x)$ *is expressed as* $\sigma(x) = Y_0(c_0, x)$, *where* c_0 *is some real constant.*

Proof. Since a zero of $\sigma(x)$ is double (cf. [1; Lemma 5.1]), the solution $\sigma(x)$ satisfies either $\sigma(x) \geq 0$ or $\sigma(x) \leq 0$ for $x > 0$. We only prove the assertion in case $\sigma(x) \geq 0$, because, if $\sigma(x) \leq 0$, using (E.$-$), we can prove in a similar way. Let $u = \phi(x)$ ($\not\equiv 0$) be a solution of (E.0) such that $\tanh^2\phi(x) = \sigma(x)$ and $\phi(x) \to 0$ as $x \to +0$. It is sufficient to show that $\phi(x)$ can be expressed as $\phi(x) = x^{\sqrt{\alpha/2}}\, w(\kappa_0, x)$ (cf. (5)) with some real constant κ_0. Substituting $\phi(x)$ into (1) and putting $x = e^{-t}$, $\rho(t) = t^{-1/2}\phi(e^{-t})$, we have

$$(6)\quad \rho''(t) + t^{-1}\rho'(t) = \left(\frac{\alpha}{2} + F_1(t)\right)\rho(t)$$

where $F_1(t) = (1/4)t^{-2} + F_0(e^{-t}, \phi(e^{-t}))$. Since $F_1(t) \to 0$ as $t \to +\infty$ (i.e. $x \to +0$), we can take a sufficiently large positive constant T_0 such that, for $t > T_0$,

$$(7)\quad |\rho(t)| < 1,$$

$$(8)\quad t^{-1}(t\rho'(t))'\, \rho(t)^{-1} \geq (\sqrt{\alpha/2} - \varepsilon)^2,$$

where $\varepsilon = \min\{1/3, \sqrt{2\alpha}/7\}$. Then we obtain

$$(9)\quad \rho(t) = O(\exp(-(\sqrt{\alpha/2} - \varepsilon)t))$$

as $t \to +\infty$, namely

$$\phi(x) = O(x^{\sqrt{\alpha/2}-\varepsilon}(\log x)^{1/2}) = O(x^{\sqrt{\alpha/2}-2\varepsilon})$$

as $x \to +0$. Estimate (9) can be derived in exactly the same way as in the proof of [1; Lemma 4.3]. In place of (3.1), (4.11) and (4.12) in [1], we use (6), (7) and (8) respectively. Since $\Psi(x) = x^{-\sqrt{\alpha/2}}\phi(x)$ satisfies equation (2), we have, for some complex constants C_1 and C_2,

$$\Psi(x) - C_1 - C_2 x^{-\sqrt{2\alpha}}$$
$$= V(x) := \frac{1}{\sqrt{2\alpha}} \int_0^x \frac{1}{\xi}\left(1 - \left(\frac{\xi}{x}\right)^{\sqrt{2\alpha}}\right) F(\xi, \Psi(\xi))\, \Psi(\xi)\, d\xi.$$

Using (3) and the estimate $\Psi(x) = O(x^{-2\varepsilon})$, we have $V(x) = O(x^{\varepsilon'})$ as

$x \to +0$, where $\varepsilon' = \min\{1-2\varepsilon, \sqrt{2\alpha} - 6\varepsilon\} > 0$. This yields $C_1 x^{2\varepsilon} + C_2 x^{2\varepsilon - \sqrt{2\alpha}} = O(1)$ as $x \to +0$, from which we derive $C_2 = 0$. Therefore $\Psi(x) = C_1 + V(x)$, where C_1 is a real constant. This implies that $\Psi(x)$ satisfies system (4) with $\kappa = C_1$, and that $\psi(x)$ can be expressed as $\psi(x) = x^{\sqrt{\alpha/2}} w(C_1, x)$ with some real constant C_1. Thus the proof is completed.

2. In case $\alpha = 0$, we have the following.

Theorem 3. *Assume that $\alpha = 0$. Then, for every $c \in \boldsymbol{R} - \{0\} \cup \{1\}$, equation* (V) *admits a solution $y_0(c, x)$ satisfying*

$$y_0(c, x) = c + O(x),$$
$$(d/dx) y_0(c, x) = O(1)$$

as $x \to +0$. Furthermore the solution $y_0(c, x)$ is a unique solution approaching c as $x \to +0$.

Proof. Assume that $0 < c < 1$. To prove the existence of the solution $y_0(c, x)$, it is sufficient to show that equation (E.0) admits a solution $v(C, x)$ satisfying

(10) $$v(C, x) = C + O(x), \quad v'(C, x) = O(1)$$

as $x \to +0$, where $C = (1/2)\log((1+\sqrt{c})/(1-\sqrt{c}))$ (i.e. $c = \tanh^2 C$). Equation (E.0) can be written in the form

(11) $$(xu')' = uG(x, u), \qquad G(x, u) = O(1),$$

if $|u - C| < 1$, $0 < x < 1$. Consider a system of integral equations of the form

(12) $$\begin{cases} v(x) = C + \displaystyle\int_0^x \frac{1}{\xi} \int_0^\xi G(t, v(t)) v(t)\, dt\, d\xi, \\ v'(x) = \displaystyle\frac{1}{x} \int_0^x G(t, v(t)) v(t)\, dt \end{cases}$$

for $0 < x < 1$, of which the solutions satisfy equation (11). By the method of successive approximation we can prove that equation (12) admits a solution $v(C, x)$ satisfying (10). Next let $\phi(x)$ be a solution of (E.0) such that $\phi(x) \to C$ as $x \to +0$. Then $\phi(x)$ satisfies

$$\phi(x) = C_1 + C_2 \log x + \int_0^x \frac{1}{\xi} \int_0^\xi G(t, \phi(t)) \phi(t)\, dt\, d\xi$$

near $x = 0$, where C_1 and C_2 are some complex constants. Since the integral in the right-hand member tends to 0 as $x \to +0$, we have $C_1 = C$ and $C_2 = 0$. Therefore $\phi(x) = v(C, x)$, which implies the second assertion of the theorem. If $c < 0$, using (E.$-$), we can prove the theorem in a similar way. Finally to treat the case where $c > 1$ we note the equation

(13) $$z'' = \left(\frac{1}{2z} + \frac{1}{z-1}\right) z'^2 - \frac{z'}{x} - \frac{\gamma z}{x} - \frac{\delta z(z+1)}{z-1},$$

which is obtained from (V) with $\alpha = 0$ by putting $y = 1/z$. Then it is easy to see that the solution $y_0(c, x)$ of (V) with $c > 1$ corresponds to that of (13) with $0 < c < 1$, from which the theorem follows immediately.

Reference

[1] Shimomura, S.: On solutions of the fifth Painlevé equation on the positive real axis. I. Funkcial. Ekvac., **28**, 341–370 (1985).