34. Generating Functions for the Spherical Functions on Some Classical Gelfand Pairs

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Introduction. Let $\boldsymbol{F}$ be $\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$ i.e. the field of real or complex numbers or of quaternions, and $x \mapsto \bar{x}$ the usual conjugation in $\boldsymbol{F}$. We define the following quadratic form in $\boldsymbol{F}^{n+1}$.

$$(x, y)_- = -\bar{x}_0 y_0 + \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n.$$

Let $U(1, n; \boldsymbol{F})$ be the group of the linear transformations g in $\boldsymbol{F}^{n+1}$ which satisfy $(gx, gy)_- = (x, y)_-$ for all $x, y \in \boldsymbol{F}^{n+1}$. We define the group G as follows.

1. If $\boldsymbol{F} = \boldsymbol{R}$, G is the connected component of the unit element in $U(1, n; \boldsymbol{R})$, i.e. $G = SO_0(1, n)$.
2. If $\boldsymbol{F} = \boldsymbol{C}$, G is the group of all the elements $g \in U(1, n; \boldsymbol{C})$ of determinant one, i.e. $G = SU(1, n)$.
3. If $\boldsymbol{F} = \boldsymbol{H}$, $G = U(1, n; \boldsymbol{H})$, i.e. $G = Sp(1, n)$.

Let $B(\boldsymbol{F}^n)$ be the unit ball in $\boldsymbol{F}^n$ and $S(\boldsymbol{F}^n)$ be the unit sphere in $\boldsymbol{F}^n$. The group G acts transitively on $B(\boldsymbol{F}^n)$ and $S(\boldsymbol{F}^n)$ as follows:

for $x = {}^t(x_1, \cdots, x_n) \in \boldsymbol{F}^n$ and $g = (g_{pq})_{0 \le p, q \le n} \in G$, we define

$$x' = gx,$$

where $x' = {}^t(x'_1, \cdots, x'_n)$, with

$$x'_p = \left(g_{p0} + \sum_{q=1}^n g_{pq} x_q\right)\left(g_{00} + \sum_{q=1}^n g_{0q} x_q\right)^{-1}, \quad 1 \le p \le n.$$

Let K be the isotropy group of $O \in B(\boldsymbol{F}^n)$ in G. Then K is a maximal compact subgroup of G and $G/K \cong B(\boldsymbol{F}^n)$. Let $G = KAN$ be the corresponding Iwasawa decomposition and M be the centralizer of A in K. Then M is the isotropy group of $e_1 = {}^t(1, 0, \cdots, 0) \in S(\boldsymbol{F}^n)$ in K and $K/M \cong S(\boldsymbol{F}^n)$ is the Martin boundary on $G/K \cong B(\boldsymbol{F}^n)$. As is well known (cf. [1], [2]), the spherical functions on K/M play an important role in the harmonic analysis on G/K.

We note:

$$K \cong \begin{cases} SO(n) \\ U(n) \\ Sp(1) \times Sp(n) \end{cases}; \quad M \cong \begin{cases} SO(n-1) & (\text{if } \boldsymbol{F} = \boldsymbol{R}) \\ U(n-1) & (\text{if } \boldsymbol{F} = \boldsymbol{C}). \\ Sp(1) \times Sp(n-1) & (\text{if } \boldsymbol{F} = \boldsymbol{H}) \end{cases}$$

Let φ be a zonal spherical function of the real case $SO(n)/SO(n-1) \cong S(\boldsymbol{R}^n)$. Then φ depends only on η_1 ($\eta = {}^t(\eta_1, \cdots, \eta_n) \in S(\boldsymbol{R}^n)$) and there exists a unique nonnegative integer p such that

$$\varphi(\eta) = C_p^{(n-2)/2}(\eta_1) / C_p^{(n-2)/2}(1), \quad \eta = {}^t(\eta_1, \cdots, \eta_n) \in S(\boldsymbol{R}^n),$$

where $C_p^{(n-2)/2}$ is the Gegenbauer polynomial. It is well known that a generating function for the Gegenbauer polynomials $C_p^{(n-2)/2}$, $p=0, 1, 2, \cdots$, is given as follows.

$$(1-2tx+t^2)^{-(n-2)/2}=\sum_{p=0}^{\infty} C_p^{(n-2)/2}(x)t^p, \quad -1\leq x\leq 1, \quad -1<t<1.$$

This formula also gives a generating function for the zonal spherical functions of $SO(n)/SO(n-1)$.

In this paper, we shall show that we can also give generating functions for spherical functions in the complex and the quaternion cases. The proof will be published elsewhere.

Suppose that $n\geq 2$ throughout this paper.

1. Complex case. Let $H_{p,q}^{(n)}$ denote the space of restrictions to $S(\boldsymbol{C}^n)$ of harmonic polynomials $f(\xi, \bar{\xi})$ on $\boldsymbol{C}^n$ which are homogeneous of degree p in ξ and degree q in $\bar{\xi}$. Then it is known (cf. [2], [3]) that $H_{p,q}^{(n)}$ is $U(n)$-irreducible and moreover $L^2(S(\boldsymbol{C}^n))=\bigoplus_{p,q=0}^{\infty} H_{p,q}^{(n)}$. Let $\varphi_{p,q}^{(n)}$ be the zonal spherical function which belongs to $H_{p,q}^{(n)}$. Then a generating function for the functions $\varphi_{p,q}^{(n)}$ is given in the following theorem.

Theorem 1. *If $w, z \in \boldsymbol{C}$, $|w|<1$, $|z|\leq 1$, then*

$$(1) \qquad (1-2\,\mathrm{Re}\,(wz)+|w|^2)^{1-n}=\sum_{p,q=0}^{\infty} a_{pq}^{(n)} Q_{pq}^{(n)}(z) w^p \bar{w}^q,$$

where

$$Q_{pq}^{(n)}(\eta_1)=\varphi_{pq}^{(n)}(\eta), \quad \eta \in S(\boldsymbol{C}^n),$$

and

$$a_{pq}^{(n)}=\frac{\Gamma(n+p-1)}{\Gamma(n-1)\Gamma(p+1)}\,\frac{\Gamma(n+q-1)}{\Gamma(n-1)\Gamma(q+1)}.$$

The series on the right hand side converges absolutely and uniformly for $|z|\leq 1$ and $|w|\leq\rho$ for each $\rho<1$.

In the formula (1), if we put $w=re^{i\theta}$, then we have

$$(2) \qquad (1-2r\,\mathrm{Re}\,(e^{i\theta}z)+r^2)^{1-n}=\sum_{p,q=0}^{\infty} a_{pq}^{(n)} Q_{pq}^{(n)}(z) e^{i(p-q)\theta} r^{p+q}.$$

This formula can be interpreted as follows.

The zonal spherical functions $\varphi_{pq}^{(n)}$ appear as the coefficients in the expansion of the left hand side of (2) by the powers of r and the spherical functions of $U(1)\cong S(\boldsymbol{R}^2)$. This interpretation for generating function will be adapted to the quaternion case.

2. Quaternion case. A zonal spherical function φ of K/M depends only on η_1, more precisely on $\mathrm{Re}\,(\eta_1)$ and $|\eta_1|$ ($\eta={}^t(\eta_1, \cdots, \eta_n) \in S(\boldsymbol{H}^n)$), and there uniquely exists a pair of nonnegative integers (p, q) such that

$$\varphi(\eta)=c_{pq} C_p^1\left(\frac{\mathrm{Re}\,(\eta_1)}{|\eta_1|}\right)|\eta_1|^p F(-q, p+q+2n-1\,;\, p+2\,;\, |\eta_1|^2),$$

where

$$c_{pq}=\frac{(-1)^q (p+2)_q}{(2(n-1))_q}[C_p^1(1)]^{-1}.$$

See Theorem 3.1 in [3], p. 144 and the formula (16) in [1], p. 170 and we follow the notations in [4]. From now on, we denote φ by $\varphi_{pq}^{(n)}$. Then a generating function for the functions $\varphi_{pq}^{(n)}$ is given in the following theorem.

Theorem 2. *If $z \in \boldsymbol{H}$, $|z| \leq 1$, $u \in Sp(1)$ and $0 \leq r < 1$, then*

$$(3)\quad \int_{Sp(1)} [1-2r\,\mathrm{Re}\,(mzm^{-1}u)+r^2]^{1-2n}dm = \sum_{p,q=0}^{\infty} \beta_{pq}^{(n)} R_{pq}^{(n)}(z) C_p^1(\mathrm{Re}\,(u)) r^{p+2q},$$

where dm is the normalized Haar measure on $Sp(1)$ and

$$R_{pq}^{(n)}(\eta_1) = \varphi_{pq}^{(n)}(\eta), \quad \eta \in S(\boldsymbol{H}^n),$$

and

$$\beta_{pq}^{(n)} = \frac{p+1}{\Gamma(p+q+2)} \frac{(2n-1)_{p+q}(2n-2)_q}{q\,!}.$$

The series on the right hand side converges absolutely and uniformly for $|z| \leq 1$, $u \in Sp(1)$ and $r \leq \rho$ for each $\rho < 1$.

The formula (3) means that the zonal spherical functions $\varphi_{pq}^{(n)}$ appear as the coefficients in the expansion of the left hand side of (3) by the powers of r and the spherical functions of $Sp(1) \cong S(\boldsymbol{R}^4)$.

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