82. Moishezon Fourfolds Homeomorphic to Q_C^4

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§ 0. **Introduction.** In general, there are many different complex manifolds having the same underlying topological or differentiable structure. However there are a few exceptional cases where we can expect that homeomorphy to a given compact complex manifold implies analytic isomorphism to it, for instance, a compact Hermitian symmetric space. Among Hermitian symmetric spaces, the complex projective space $\boldsymbol{P}_C^n$ and a smooth hyperquadric $\boldsymbol{Q}_C^n$ in $\boldsymbol{P}_C^{n+1}$ seem to be nice exceptions which we can handle with algebraic methods. In [6] we studied the complex projective space $\boldsymbol{P}_C^n$, while we study a smooth hyperquadric $\boldsymbol{Q}_C^n$ in $\boldsymbol{P}_C^{n+1}$ in [7]. A goal we have in mind is the following

Conjecture MQ_n. *Any Moishezon complex manifold homeomorphic to $\boldsymbol{Q}_C^n$ is isomorphic to $\boldsymbol{Q}_C^n$.*

The conjecture has been solved partially by Brieskorn [1] under the assumption that the manifold in question is *Kählerian* and odd-dimensional. In the even-dimensional *Kählerian* case, there still remains a possibility of manifolds of general type. Recently Kollár [2] and the author [4] solved Conjecture MQ_3 in the affirmative, each supplementing the other. Peternell [8] [9] also asserts the same consequence.

Theorem. *Any Moishezon threefold homeomorphic to $\boldsymbol{Q}_C^3$ is isomorphic to $\boldsymbol{Q}_C^3$.*

The purpose of the present article is to report a partial solution [7] to the above conjecture in dimension 4. We also report some results on threefolds with the first Chern class divisible by three and possibly with the second Betti number b_2 greater than one.

§ 1. **A complete intersection l_V.** (1.1) Let X be a complete nonsingular algebraic variety (or a compact complex manifold) of dimension n, L a line bundle on X. We assume $h^0(X, L) \geq n$. Let V be an $(n-1)$-dimensional subspace of $H^0(X, L)$, $l := l_V$ a scheme-theoretic complete intersection associated with V. This means that the ideal I_l of O_X defining l is defined by $I_l = \sum_{s \in V} sO_X$. Let $B := \mathrm{Bs}|L|$, the base locus of $|L|$. We say that C is *a reduced curve-component of* l if C is an irreducible one-dimensional component of l along which l is reduced generically.

(1.2) **Lemma.** *Assume $c_1(X) = nc_1(L)$, and $h^0(X, L) \geq n$ and let l be a scheme-theoretic complete intersection of $(n-1)$-members of $|L|$. Assume moreover that l has a reduced curve-component C with $LC \geq 1$ outside B. Then one of the following cases occurs.*

(1.2.1) *$LC=2$, $C\simeq \boldsymbol{P}^1$, $N_{C/X}\simeq O_C(2)^{\oplus(n-1)}$, C is a connected component of l.*

(1.2.2) *$LC=1$, $C\simeq \boldsymbol{P}^1$, $N_{C/X}\simeq O_C\oplus O_C(1)^{\oplus(n-2)}$, and C intersects B at a point p transversally, where*

$$I_{l,p}=(x_1,\cdots,x_{n-2},x_{n-1}x_n),$$
$$I_{C,p}=(x_1,\cdots,x_{n-2},x_{n-1}),$$
$$I_{B,p}=(x_1,\cdots,x_{n-2},x_n)$$

by choosing a suitable local coordinate $x_1,\cdots,x_n$ at p.

(1.2.3) *There is another component C_1 of l such that $C_i\simeq \boldsymbol{P}^1$, $C=C_0$, $LC_i=1$, $N_{C_i/X}\simeq O_{C_i}\oplus C_{C_i}(1)^{\oplus(n-2)}$ $(i=0,1)$. The components C_0 and C_1 intersect transversally at a point p where*

$$I_{l,p}=(x_1,\cdots,x_{n-2},x_{n-1}x_n),$$
$$I_{C_0,p}=(x_1,\cdots,x_{n-2},x_{n-1}),$$
$$I_{C_1,p}=(x_1,\cdots,x_{n-2},x_n),$$
$$I_{B,p}=(x_1,\cdots,x_{n-2},x_{n-1},x_n)$$

in terms of suitable coordinates at p.

(1.2.4) *There is a chain of m (≥ 1) smooth rational curves C_i $(0\leq i\leq m)$ such that*

$$C=C_0,\quad LC_0=LC_m=1,\quad LC_i=0\quad (1\leq i\leq m-1)$$
$$N_{C_i/X}\simeq\begin{cases} O_{C_i}\oplus O_{C_i}(1)^{\oplus(n-2)} & (i=0,m)\\ O_{C_i}(-2)\oplus O_{C_i}^{\oplus(n-2)} \text{ or } O_{C_i}(-1)^{\oplus 2}\oplus O_{C_i}^{\oplus(n-3)} & (1\leq i\leq m-1).\end{cases}$$

The curves C_j and C_i $(j<i)$ do not intersect unless $j=i-1$, while C_{i-1} and C_i intersect transversally at a point p_i where

$$I_{l,p_i}=(x_1,\cdots,x_{n-2},x_{n-1}x_n),$$
$$I_{C_{i-1},p_i}=(x_1,\cdots,x_{n-2},x_{n-1}),$$
$$I_{C_i,p_i}=(x_1,\cdots,x_{n-2},x_n)$$

in terms of suitable local coordinates at p_i. Moreover $C_0+\cdots+C_m$ is a connected component of l with $C_i\cap B_{\mathrm{red}}=\phi$ $(1\leq i\leq m-1)$.

§ 2. Moishezon fourfold homeomorphic $\boldsymbol{Q}^4$.

(2.1) **Lemma.** *Let X be a Moishezon manifold of dimension n with $b_2(X)=1$, L a line bundle on X. Assume that $c_1(X)=nc_1(L)$ and $h^0(X,O_X(L))\geq n+1$. If a complete intersection of general $(n-1)$-members of $|L|$ has an irreducible curve-component C with $LC\geq 2$ outside $\mathrm{Bs}|L|$, then $X\simeq \boldsymbol{Q}^n$.*

(2.2) **Lemma.** *Let X be a Moshezon 4-fold homeomorphic to $\boldsymbol{Q}^4$, and L a line bundle on X with $L^4=2$. Assume that $h^0(X,L)\geq 2$. Let D and D' be distinct members of $|L|$, τ the scheme-theoretic complete intersection $D\cap D'$. Then τ is pure two-dimensional Gorenstein and we have*

(2.2.1) $\mathrm{Pic}\,X\simeq \boldsymbol{Z}L$, $K_X\simeq -4L$,

(2.2.2) $H^p(X,-qL)=0$ $(p=0, q\geq 1$, *or* $1\leq p\leq 3$, $0\leq q\leq 4$, or $p=4$, $q\leq 3)$

(2.2.3) $H^p(\tau,-qL_\tau)=0$ $(p=0, q=1,2$, *or* $p=1$, $0\leq q\leq 2$, or $p=2$, $q=0,1)$

(2.2.4) $H^0(X,O_X)\simeq H^0(D,O_D)\simeq H^0(\tau,O_\tau)\simeq \boldsymbol{C}$, *and* $|L|_\tau=|L_\tau|$.

(2.3) **Theorem.** *Let X be a Moishezon 4-fold homeomorphic to $\boldsymbol{Q}^4$, and L a line bundle on X with $L^4=2$. Assume that $h^0(X,L)\geq 5$. Then $X\simeq \boldsymbol{Q}^4$.*

(2.4) **Corollary.** *Any global deformation of Q^4 is isomorphic to Q^4.*

§ 3. **Moishezon threefolds with c_1 divisible by 3.** (3.1) **Theorem.** *Let X be a Moishezon 3-fold and L a line bundle on X with $L^3\geq 1$. Assume that $h^1(X, O_X)=0$, $c_1(X)=3c_1(L)$, $h^0(X, L)\geq 2$, and* $\dim \mathrm{Bs}\,|L|\leq 1$. *Then $X\simeq Q^3$ or $P(\mathcal{F}(a, b, 0))$ $(a\geq b\geq n\geq 0,\ a+b=3n+2)$, where $\mathcal{F}(a, b, 0):= O_{P^1}(a)\oplus O_{P^1}(b)\oplus O_{P^1}$.*

(3.2) We assume a Moishezon 3-fold X to have line bundles L and F such that

$$\mathrm{Pic}\,X\simeq H^2(X, Z)\simeq ZL\oplus ZF,\quad H^4(X, Z)\simeq ZL^2\oplus ZLF,$$
$$c_1(X)=3c_1(L),\quad c_2(X)=3L^2+2LF,\quad L^3=2,\quad L^2F=1,\quad F^2=0,$$
$$h^q(X, O_X)=0\ (q\geq 1),\quad h^0(X, L-F)\geq 2,\quad h^0(X, F)\geq 2.$$

(3.3) **Theorem.** *Let X be a Moishezon 3-fold. If X satisfies the conditions in* (3.2), *then $X\simeq P(\mathcal{F}(a, b, 0))$ for some $a\geq b\geq 0$ and $a+b\equiv 2 \bmod 3$.*

(3.4) **Theorem.** *Let X be a Moishezon 3-fold homeomorphic to $P(\mathcal{F}(2, 0, 0))$. If $h^0(X, L-2F)\geq 1$ and $h^0(X, F)\geq 2$, then $X\simeq P(\mathcal{F}(a, b, 0))$ for some $a\geq b\geq 0$, $a+b\equiv 2 \bmod 3$, $(a, b)\neq(1, 1)$.*

Table. Threefolds with $h^1(X, O_X)=0$, $c_1(X)=3c_1(L)$, $L^3\geq 1$, $h^0(X, L)\geq 2$

	$\mathrm{Bs}\,\|L\|$	$\dim W$*	Sing W	X
1	ϕ	3	ϕ	Q^3
2	ϕ	3	one point	$P(\mathcal{F}(1, 1, 0))$
3	ϕ	3	P^1	$P(\mathcal{F}(2, 0, 0))$
4	curve	2	at most one point	$P(\mathcal{F}(a, b, 0))^{a\geq b\geq n\geq 1}_{a+b=3n+2}$
5	surface	?	?	?

* W is the image of the rational map $h: X\to P^m$ associated with $|L|$, $m=h^0(X, L)-1$.

§ 4. **Global deformations of $(\mathcal{F}(a, b, 0))$.** (4.1) Let $k=0$ or 1. We assume that a Moishezon 3-fold X has line bundles L and F satisfying the following conditions,

$$(4.1.k)\qquad \mathrm{Pic}\,X\simeq H^2(X, Z)\simeq ZL\oplus ZF,\quad H^4(X, Z)\simeq ZL^2\oplus ZLF,$$
$$c_1(X)=3L+(2-k)F,\quad L^3=k,\quad L^2F=1,\quad F^2=0,$$
$$h^q(X, O_X)=0\ (q\geq 1),\quad h^0(X, L)\geq 3,$$
$$h^0(X, L-F)\geq 1,\quad h^0(X, F)\geq 2,\quad \chi(X, -L)=0.$$

(4.2) **Theorem.** *Let $k=0$ or 1. If a Moishezon 3-fold X satisfies the condition* (4.1.k), *then $X\simeq P(\mathcal{F}(a, b, 0))$ for some $a\geq b\geq 0$, $a+b\geq 1$ and $a+b\equiv k \bmod 3$.*

We note that (8.4) does not classify all the global deformations of $P(\mathcal{F}(a, b, 0))$ with $a+b\equiv 0 \bmod 3$, because $h^0(X, L-F)=0$ is possible. Combining (4.2) with (3.3), we infer

(4.3) **Theorem.** *Let* $k=1$ *or* 2. *The set of all* $\boldsymbol{P}^2$*-bundles* $\boldsymbol{P}(\mathscr{F}(a, b, 0))$ *over* $\boldsymbol{P}^1$ *with* $a \geq b \geq 0$, $a+b \equiv k \bmod 3$ *is stable under global deformation.*

References

[1] E. Brieskorn: Ein Satz über die komplexen Quadriken. Math. Ann., **155**, 184–193 (1964).

[2] J. Kollár: Flips, flops, minimal models etc. (1990) (preprint).

[3] I. Nakamura: Moishezon threefolds homeomorphic to $\boldsymbol{P}^3$. J. Math. Soc. Japan, **39**, 521–535 (1987).

[4] ——: Threefolds homeomorphic to a hyperquadric in $\boldsymbol{P}^4$. Algebraic Geometry and Commutative Algebra in Honor of M. Nagata. Kinokuniya, Tokyo, Japan, pp. 379–404 (1987).

[5] ——: A subadjunction formula and Moishezon fourfolds homeomorphic to $\boldsymbol{P}^4_{\boldsymbol{C}}$. Proc. Japan Acad., **67**A, 65–67 (1991).

[6] ——: On Moishezon manifolds homeomorphic to $\boldsymbol{P}^n_{\boldsymbol{C}}$ (1991) (preprint).

[7] ——: Moishezon fourfolds homeomorphic to $\boldsymbol{Q}^4_{\boldsymbol{C}}$ and Moishezon-Fano threefolds of index three (1991) (preprint).

[8] T. Peternell: A rigidity theorem for $\boldsymbol{P}_3(\boldsymbol{C})$. Manuscripta Math., **50**, 397–428 (1985).

[9] ——: Algebraic structures on certain 3-folds. Math. Ann., **274**, 133–156 (1986).