57. A Reduction of Hamiltonian Systems with Multi-time Variables Along a Regular Singularity

By Hironobu KIMURA,*) Atusi MATUMIYA,**) and Kyoichi TAKANO**)

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1. Introduction. Let $(t, x)=(t_1, \cdots, t_N, x_1, \cdots, x_{2n})$ be the coordinates of $\mathbf{C}^{2n+N}$ and let $D(r, \rho)$ be an unbounded domian in $\mathbf{C}^{2n+N}$ defined by

$$D(r, \rho):=\{(t, x) \in \mathbf{C}^{2n+N}; |t|<r, |x_1 x_{n+1}|, |t_1 x_{n+1}|, |x_i|<\rho, (i \neq n+1)\}$$

where $|a|:=\max\{|a_1|, \cdots, |a_m|\}$ for $a=(a_1, \cdots, a_m) \in \mathbf{C}^m$. The projection image to $D(r, \rho)$ to the t-space is a polydisk with center 0, which we denote by $\Delta(r):=\{t \in \mathbf{C}^N; |t|<r\}$. The domain $D(r, \rho)$ is a neighbourhood of $(0, 0)$.

Consider a completely integrable Hamiltonian system of the form:

$$(1) \qquad t_i \partial_i x = J H_x^i, \qquad J=\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \qquad 1 \leq i \leq N$$

with Hamiltonians $H^1, \cdots, H^N$ holomorphic in $D(r, \rho)$, where $\partial_i = \partial/\partial t_i$ and $H_x^i := {}^t(H_{x_1}^i, \cdots, H_{x_{2n}}^i)$ is the gradient vector of H^i in x. The system (1) is said to have *a singularity of regular type along a hyperplane* $S:=\{t \in \Delta(r); t_1=0\}$, if H^i/t_i $(2 \leq i \leq N)$ are holomorphic in $D(r, \rho)$ and if H^1 does not have t_1 as a factor.

The purpose of this note is to obtain a reduction theorem for the system (1) with a singularity of regular type along S (Theorem 1). This result will be applied to the Hamiltonian system $\mathcal{H}_n$ (see § 2) which is a generalization of the sixth Painlevé system [7] to a system of partial differential equations obtained by a monodromy preserving deformation.

We say that a symplectic transformation $\phi: (t, x) \rightarrow (t, X)$ is *♯-symplectic* if ϕ is holomorphic on $D(r, \rho)$ and if $D(r', \rho') \subset \phi(D(r, \rho))$ for some positive r' and ρ'.

We define a class of Hamiltonians studied in this note. Consider a Hamiltonian system (1) with a Hamiltonian $\mathbf{H}=(H^1, \cdots, H^N)$. We expand H^i in x as

$$H^i(t, x) = {}^t H_x^i(t, 0) x + \frac{1}{2} {}^t x H_{xx}^i(t, 0) x + \sum_{\substack{\alpha+e_1+e_{n+1} \geq 0 \\ |\alpha+e_1+e_{n+1}| \geq 3}} h_\alpha^i(t) x^{\alpha+e_1+e_{n+1}}$$

for $1 \leq i \leq N$, where H_{xx}^i denotes the Hessian of H^i with respect to x and $x^{\alpha+e_1+e_{n+1}} = x_1^{\alpha_1+1} \cdots x_n^{\alpha_n} x_{n+1}^{\alpha_{n+1}+1} \cdots x_{2n}^{\alpha_{2n}}$.

We assume the following four conditions:

(A-1) $H^1, H^2/t_2, \cdots, H^N/t_N$ are bounded holomorphic functions in $D(r, \rho)$.

(A-2) H^1 satisfies

*) Department of Mathematics, College of Arts and Sciences, University of Tokyo.
**) Department of Mathematics, Faculty of Sciences, Kobe University.

$$JH^1_x|_{t_1=0,x=0}=\begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad JH^1_{xx}|_{t_1=0,x=0}=\left[\begin{array}{c:c} \begin{matrix}\eta & \\ * & 0 \\ \vdots & \end{matrix} & 0 \\ \hdashline \begin{matrix} * \cdots * \\ \vdots \quad 0 \\ * \end{matrix} & \begin{matrix} -\eta * \cdots * \\ 0 \end{matrix} \end{array}\right]$$

where $*$ stands for a function of $t'=(t_2, \cdots, t_N)$.

(A-3) $\eta \in \mathbf{C}\backslash(-\infty, 0]\cup[1, \infty)$.

(A-4) $h^1_\alpha(t)|_{t_1=0}=0$ if $\alpha_1=\alpha_{n+1}$ and $\alpha \notin \mathbf{Z}(e_1+e_{n+1})$.

The condition (A-1) implies that the singular locus of the system (1) is $S=\Delta(r)\cap\{t_1=0\}$. Set

$$\mathcal{A}_{r,\rho}:=\{\mathbf{H}=(H^1, \cdots, H^N)\,;\ \mathbf{H} \text{ satisfies } (A-1), \cdots, (A-4)\}.$$

Then our main theorem is

Theorem 1. *For a completely integrable Hamiltonian system* (1) *with* $\mathbf{H}\in\mathcal{A}_{r,\rho}$, *there exists a* $\sharp$*-symplectic transformation* $(t, x)\to(t, X)$ *given by* $x=\varphi(t, X)\in\mathcal{O}^{2n}_{D(r',\rho')}$, r', $\rho'>0$, *such that it takes the system* (1) *into*

$$t_i\partial_i X=JH^i_{\infty,X}, \qquad 1\le i\le N \tag{2}$$

with the Hamiltonian $\mathbf{H}_\infty$:

$$H^1_\infty=\eta X_1X_{n+1}+\sum_{m\ge1} h^1_{m(e_1+e_{n+1})}(0)(X_1X_{n+1})^{m+1},$$
$$H^2_\infty=\cdots=H^N_\infty=0.$$

Remark that we can obtain the Hamiltonian $\mathbf{H}_\infty$ from a given Hamiltonian $\mathbf{H}\in\mathcal{A}_{r,\rho}$ by picking up the terms with powers $\alpha=m(e_1+e_{n+1})$ and by setting $t=0$. Moreover, by solving the system (2), we can obtain a general solution of the system (1) through the $\sharp$-symplectic transformation. In fact,

Corollary 2. *A Hamiltonian system* (1) *with* $\mathbf{H}\in\mathcal{A}_{r,\rho}$ *has a* $2n$*-parameter family of solutions of the form*

$$x(t)=\varphi(t, X(t)),$$

where $\varphi(t, X)$ *is the transformation given in Theorem* 1 *and*

$$X(t)=(c_1t_1^{\eta+h(c_1c_{n+1})}, c_2, \cdots, c_n, c_{n+1}t_1^{-\eta-h(c_1c_{n+1})}, c_{n+2}, \cdots, c_{2n}),$$
$$h(z)=\sum_{m\ge1}(m+1)h^1_{m(e_1+e_{n+1})}(0)z^m,$$

$c_1, \cdots, c_{2n}$ *being complex constants.*

2. The Hamiltonian system $\mathcal{H}_n$. The system $\mathcal{H}_n$ is a completely integrable system of exterior differential 1-forms

$$\mathcal{H}_n: \quad \begin{cases} \omega_i=dq_i-\sum\limits_{1\le j\le n}\dfrac{\partial H^j}{\partial p_i}dt_j=0, \\ \omega_{n+i}=dp_i+\sum\limits_{1\le j\le n}\dfrac{\partial H^j}{\partial q_i}dt_j=0, \end{cases} \quad (1\le i\le n).$$

The Hamiltonians H^i are polynomials in (q, p) with coefficients rational in $t=(t_1, \cdots, t_n)$ of the form

$$H^i=\frac{1}{t_i(t_i-1)}\Big[\sum_{1\le j,k\le n}E^i_{jk}(t, q)p_jp_k-\sum_{1\le j\le n}F^i_j(t, q)p_j+\kappa q_i\Big]$$

with $E^i_{jk}(t, q), F^i_j(t, q) \in \mathbf{C}(t)[q]$ such that

$$E^i_{jk}=E^j_{ik}=E^k_{ij}, \qquad F^i_j=F^j_i, \qquad 1\le i, j\le n.$$

Explicitly,

$$E^i_{jk}=\begin{cases} q_i q_j q_k & \text{if } i,\ j,\ k \text{ are distinct,}\\ q_i q_j(q_j-R_{ji}) & \text{if } i\neq j=k,\\ q_i(q_i-1)(q_i-t_i)-\sum_{1\le a\le n, a\neq i} T_{ia}q_i q_a & \text{if } i=j=k,\end{cases}$$

$$F^i_j=\begin{cases} (\theta_{n+1}-1)q_i(q_i-1)+\theta_{n+2}q_i(q_i-t_i)+\theta_i(q_i-1)(q_i-t_i) & \\ \qquad +\sum_{1\le k\le n, k\neq i}\{\theta_k q_i(q_i-R_{ik})-\theta_i T_{ik}q_k\} & \text{if } i=j,\\ (\sum_{1\le k\le n+2}\theta_k-1)q_i q_j-\theta_i R_{ij}q_j-\theta_j R_{ji}q_i & \text{if } i\neq j,\end{cases}$$

where

$$\kappa=\frac{1}{4}[(\sum_{1\le i\le n+2}\theta_i-1)^2-\theta^2_{n+3}],$$

$$R_{ij}=\frac{t_i(t_j-1)}{t_j-t_i}, \qquad T_{ij}=\frac{t_i(t_i-1)}{t_i-t_j}$$

and $\theta_1, \cdots, \theta_{n+3}$ are complex constants.

Let $V\simeq \mathbf{C}^{n+3}$ be the space of parameters $\theta:=(\theta_1, \cdots, \theta_{n+3})$ of $\mathcal{H}_n$, and let $\mathcal{H}_n(\theta)$ be the system $\mathcal{H}_n$ with a parameter $\theta\in V$. For a birational transformation $T:(q,p,t)\to(q^*,p^*,t^*)$, we denote by $T\cdot\mathcal{H}_n(\theta)$ the system

$$(T^{-1})^*\omega_i=0, \qquad 1\le i\le 2n.$$

A *symmetry* of $\mathcal{H}_n$ is a pair $\sigma:=(T, l)$ of a birational transformation $T:(q,p,t)\to(q^*,p^*,t^*)$ and an affine transformation $l: V\to V$ such that $T\cdot\mathcal{H}_n(\theta)=\mathcal{H}_n(l(\theta))$ for all $\theta\in V$. For symmetries $\sigma=(T,l)$ and $\sigma'=(T',l')$, the product and the inverse are defined by $\sigma\cdot\sigma':=(T\circ T', l\circ l')$ and $\sigma^{-1}:=(T^{-1}, l^{-1})$, respectively.

Then we have

Proposition 3. *There is a group of symmetries G of $\mathcal{H}_n$ which is isomorphic to the symmetric group $\mathbf{S}_{n+3}$ on $n+3$ elements.*

As to the explicit form of generators of G, see [5].

Consider the system $\mathcal{H}_n$ on the space $(\mathbf{P}^1)^n\times\mathbf{C}^{2n}\ni(t,q,p)$, then the singular locus S of $\mathcal{H}_n$ is

$$S=\bigcup_{1\le i,j\le n+3}S_{ij}, \qquad S_{ij}:=\{t\in(\mathbf{P}^1)^n\,;\,t_i=t_j\},$$

where $t_{n+1}=0$, $t_{n+2}=1$ and $t_{n+3}=\infty$. Set

$$S^\circ:=\bigcup_{i,j}S^\circ_{ij}, \qquad S^\circ_{ij}:=S_{ij}\setminus\bigcup_{(k,l)\neq(i,j)}(S_{ij}\cap S_{kl}),$$

and $S_{sing}=S\setminus S^\circ$. The hyperplanes S_{ij} in $(\mathbf{P}^1)^n$ are irreducible components of S and S° is the set of its smooth points. Each element $\sigma=(T,l)\in G$ induces a birational transformation $t\to t^*$ of $(\mathbf{P}^1)^n$. If there is no fear of confusion, we denote the birational transformation $t\to t^*$ also by T. We investigate how the group G acts on the singular locus S.

Proposition 4. *Let $\sigma=(T,l)$ be an element of G.*

(a) *T maps S into itself.*

(b) *If $T(S^\circ_{ij})\subset S^\circ$, T is biholomorphic on S°_{ij}.*

(c) *For any S°_{ij}, there is an element of G which induces a biholomorphic map from* $S^{\circ}_{1,n+1}$ to S°_{ij}.

By virtue of this proposition we have only to study the solutions of $\mathcal{H}_n$ along $S^{\circ}_{1,n+1}$ in order to study those along S°.

3. Restriction of $\mathcal{H}_n(\theta)$ to $S^{\circ}_{1,n+1}$. In this section we show that the restriction of $\mathcal{H}_n$ to Σ_0 (see Proposition 6) is $\mathcal{H}_{n-1}$. We make use of this fact when we apply Theorem 1 to $\mathcal{H}_n(\theta)$. Consider, in general, a completely integrable Pfaffian system

$$t_i \partial_i x = F^i(t, x), \qquad 1 \leq i \leq N \tag{3}$$

with independent variables $t=(t_1, \cdots, t_N)$ and unknowns $x={}^t(x_1, \cdots, x_p)$. Assume that

$$F^1, F^2/t_2, \cdots, F^N/t_N \in \mathcal{O}^p_U,$$

where $U=\{(t, x); |t|<r, |x|<\rho\}$. Then the system (3) has a singularity along the hyperplane $S:=\{t \in \Delta(r); t_1=0\}$. We want to find "*a Pfaffian system obtained by the restriction of the system* (3) *to its singular locus* S". To this end, suppose that there is a solution of (3) of the form

$$x=\vec{a}(t)=\sum_{m\geq 0} \vec{a}_m(t')t_1^m,$$

holomorphic at $t=0$, where $t'=(t_2, \cdots, t_N)$. Since $\lim_{t_1\to 0}\vec{a}(t)=\vec{a}_0(t')$, $\vec{a}_0(t')$ must satisfy the equations

$$F^1(0, t', x)=0, \tag{4}$$

$$t_i \partial_i x = F^i(0, t', x), \qquad 2 \leq i \leq N. \tag{5}$$

The system (5) with (4) is called *the restriction of* (3) *to its singular locus* S. Put $\Sigma=\{(0, t', x) \in U; F^1(0, t', x)=0\}$. For the restriction (4) and (5), we can prove

Proposition 5. (a) *If the system* (3) *is completely integrable, so is the system* (5).

(b) *Let $x(t')$ be a solution of the system* (5) *satisfying* $(0, t'_0, x(t'_0)) \in \Sigma$ *for some t'_0. Then $(0, t', x(t')) \in \Sigma$ as long as $x(t')$ is defined.*

Now we study the restriction of the system $\mathcal{H}_n(\theta)$ to a singular locus $S^{\circ}_{1,n+1}$. Note that the Hamiltonian H^1 has a simple pole along $S^{\circ}_{1,n+1}$ and $H^2, \cdots, H^n$ are holomorphic there. Set

$$L^1:=t_1 H^1|_{t_1=0}, \quad L^i:=H^i|_{t_1=0} \quad (2 \leq i \leq n),$$

and define the variety $\Sigma \subset \mathbf{C}^{3n}$ for $\mathcal{H}_n(\theta)$ by

$$\Sigma=\{(0, t', q, p) \in \mathbf{C}^{3n}; L^1_{q_i}=L^1_{p_i}=0 \ (1 \leq i \leq n)\}.$$

Proposition 6. *For the system $\mathcal{H}_n(\theta)$ with $1-\theta_1-\theta_{n+1}, \theta_2, \cdots, \theta_n \neq 0$, the algebraic variety Σ is decomposed into irreducible components as*

$$\Sigma=\Sigma_0 \cup \bigcup_{1 \leq i \leq 2^n} \Sigma_i,$$

where

$$\Sigma_0=\{(0, t', q, p) \in \mathbf{C}^{3n}; q_1=0, (1-\theta_1-\theta_{n+1})p_1=f(q', p')\}$$

with

$$f=\sum_{j,k\neq 1}(q_j p_j)(q_k p_k)-\sum_{k\neq 1} q_k p_k^2-\sum_{k\neq 1}(\theta q_k-\theta_k)p_k+\kappa,$$

$$\theta=\theta_1+\cdots+\theta_{n+2}-1$$

and Σ_i $(1 \leq i \leq 2^n)$ *are* $n-1$ *dimensional manifolds defined by the equations* $q=c(i)$ *and* $p=d(i)$, $c(i)$ *and* $d(i)$ *being certain constants.*

Corollary 7. *If there is a solution* $(q(t), p(t))$ *of* $\mathcal{H}_n(\theta)$ *holomorphic at* $t_0 \in S^\circ_{1,n+1}$, *then* $(t, q(t), p(t)) \in \Sigma$ *for* $t \in S^\circ_{1,n+1}$.

Consider the restriction of the system $\mathcal{H}_n(\theta)$ to $S^\circ_{1,n+1}$. By the explicit form of L^i, we see that $L^i_{q_j}|_{\Sigma_0}$ and $L^i_{p_j}|_{\Sigma_0}$ $(2 \leq i, j \leq n)$ do not contain p_1 explicitly. Therefore the system

$$(6) \qquad \frac{\partial q_j}{\partial t_i} = \frac{\partial L^i}{\partial p_j}, \quad \frac{\partial p_j}{\partial t_i} = -\frac{\partial L^i}{\partial q_j}, \quad 2 \leq i, j \leq n$$

on Σ_0 is completely integrable by virtue of Proposition 5.

Proposition 8. *If* $1-\theta_1-\theta_{n+1} \neq 0$, *the system* (6) *on* Σ_0 *is the Hamiltonian system* $\mathcal{H}_{n-1}(\theta_2, \cdots, \theta_n, \theta_1+\theta_{n+1}, \theta_{n+2}, \theta_{n+3})$.

This observation combined with Proposition 5 leads to

Proposition 9. *Suppose that* $1-\theta_1-\theta_{n+1} \notin \mathbf{Z}$. Let $(0, t'_0) \in S^\circ_{1,n+1}$ *and let* $(q', p') = (b_2(t'), \cdots, b_n(t'), b_{n+2}(t'), \cdots, b_{2n}(t'))$ *be an arbitrary solution of the system* (6), *which is* $\mathcal{H}_{n-1}(\theta_2, \cdots, \theta_n, \theta_1+\theta_{n+1}, \theta_{n+2}, \theta_{n+3})$, *holomorphic at* $t'=t'_0$. *If we define* $b_1(t')$ *and* $b_{n+1}(t')$ *so that* $(0, t', \vec{b}(t')) \in \Sigma_0$, $\vec{b}(t') := {}^t(b_1(t'), \cdots, b_{2n}(t'))$ *by using Proposition* 6, *then there is a unique solution* $(q, p) = \vec{a}(t)$ *of* $\mathcal{H}_n(\theta)$ *holomorphic in an open neighbourhood of* t_0 *in* $(\mathbf{P}^1)^n$ *satisfying*

$$\lim_{t_1 \to 0} \vec{a}(t) = \vec{b}(t').$$

In particular, we have

Corollary 10. *Suppose the same assumption as in Proposition* 9. *Then, for any* $t_0 \in S^\circ_{1,n+1}$ *and* $b \in \mathbf{C}^{2n}$ *with* $(t_0, b) \in \Sigma_0$, *there is a unique solution* $\vec{a}(t)$ *of* $\mathcal{H}_n(\theta)$ *which is holomorphic at* t_0 *and* $\lim_{t \to t_0} \vec{a}(t) = b$.

4. Application of Theorem 1 to $\mathcal{H}_n$. In this section, we use $x = (x_1, \cdots, x_{2n})$ instead of (q, p). Suppose that $\eta := 1-\theta_1-\theta_{n+1} \in \mathbf{C} \backslash (-\infty, 0] \cup [1, \infty)$, then the system $\mathcal{H}_n(\theta)$ satisfies the assumptions (A-1), $\cdots$, (A-4) in Section 1. Theorem 1 tells us that there is a $\sharp$-symplectic transformation $(t, x) \to (t, X)$ which reduces $\mathcal{H}_n(\theta)$ to a Hamiltonian system (2) with the Hamiltonian $\mathbf{H}_\infty$:

$$H^1_\infty = \eta X_1 X_{n+1} + (X_1 X_{n+1})^2, \quad H^2_\infty = \cdots = H^n_\infty = 0.$$

This observation combined with Proposition 9 leads to

Theorem 11. *Suppose that* $\eta := 1-\theta_1-\theta_{n+1} \in \mathbf{C} \backslash (-\infty, 0] \cup [1, \infty)$. *Then, for any holomorphic solution* $\vec{a}(t)$ *of* $\mathcal{H}_n(\theta)$ *at* $t_0 \in S^\circ_{1,n+1}$ *obtained in Proposition* 9, *there is a* $2n$*-parameter family of solutions of* $\mathcal{H}_n(\theta)$ *of the form*

$$x(t) = \vec{a}(t) + \varphi(t, X(t)), \quad \varphi(t, X) \in (\mathcal{M}_X)^{2n},$$

where $x = \vec{a}(t) + \varphi(t, X)$ *is a* $\sharp$*-symplectic transformation for a domain*

$$D(t_0, r, \rho) := \{(t, x) \in \mathbf{C}^{3n} \,;\, |t-t_0| < r, |x_1 x_{n+1}|, |t_1 x_{n+1}|, |x_i| < \rho \ (i \neq n+1)\}$$

with some positive constants r *and* ρ, *and* $X(t)$ *is given by*

$$X(t) = (c_1 t_1^{\eta + 2c_1 c_{n+1}}, c_2, \cdots, c_n, c_{n+1} t_1^{-\eta - 2c_1 c_{n+1}}, c_{n+2}, \cdots, c_{2n}),$$

$c_1, \cdots, c_{2n}$ *being arbitrary constants.*

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