88. *On a Certain Invariant of a Finite Unitary Reflection Group*

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1. Observations. Let V be a complex vector space of dimension l. An element $s \in \mathrm{GL}(V)$ is called a *reflection* if it has finite order and its fixed point set $\mathrm{Fix}(s)$ is a hyperplane, i.e., one-codimensional subspace. A *unitary reflection group* is a finite subgroup of $\mathrm{GL}(V)$ generated by reflections. It is known that one may choose a G-invariant Hermitian form on V. Let G be a unitary reflection group. Let $\mathcal{A}$ be the set of reflecting hyperplanes $H=\mathrm{Fix}(s)$ where $s \in G$ is a reflection. We call $\mathcal{A}$ the *arrangement defined by* G. Denote the cardinality of $\mathcal{A}$ by n.

Definition 1.1.

$$t=t(G)=\frac{2n}{l}.$$

The irreducible unitary reflection groups were completely classified by G.C. Shephard and J.A. Todd [4]. From their table we have the following

Observation 1.2. *For an irreducible unitary reflection group G, the number t is a positive integer.*

Let $\mathcal{A}$ be the arrangement defined by G. Fix $H \in \mathcal{A}$. Define the *restriction* of $\mathcal{A}$ to H by

$$\mathcal{A}''=\{H \cap K \mid K \in \mathcal{A},\ K \neq H\}.$$

Then $\mathcal{A}''$ is a finite collection of hyperplanes in H. Let n'' denote the cardinality of $\mathcal{A}''$. The restrictions $\mathcal{A}''$ was studied by P. Orlik and L. Solomon [2]. From the study we observe

Observation 1.3. *For an irreducible unitary reflection group G, we have*

$$t=n-n''+1.$$

This observation is, of course, stronger than the previous one 1.2. In this note we will give a proof of Observation 1.3 without using classification.

When G is an irreducible unitary reflection group defined over the real number field, G is a Coxeter group. In this case, the number t is equal to the Coxeter number and Observation 1.3 was proved in [3, Theorem 3.7] without using classification.

2. The main results. Let G be a unitary reflection group acting on a complex vector space V of dimension l. Choose a G-invariant Hermitian form $(\ ,\ )$ on V.

Let $\mathcal{A}$ be the arrangement defined by G. Let $K \in \mathcal{A}$. We denote the orthogonal complement of K by $K^\perp$. Then $K^\perp$ is of dimension one. Choose $\alpha_K \in K^\perp$ such that $|\alpha_K|^2=(\alpha_K, \alpha_K)=1$.

In the rest of this note we assume that G is an irreducible unitary reflection group.

Proposition 2.1. *Let $t=2n/l$. For any $v \in V$, we have*

$$2 \sum_{K \in \mathcal{A}} (v, \alpha_K)\alpha_K = tv.$$

Proof. Define a map $f: V \to V$ by

$$f(v)=\sum_{K \in \mathcal{A}} (v, \alpha_K)\alpha_K.$$

Then f is $\mathbf{C}$-linear. Note that the definition does not depend upon choice of α_K ($K \in \mathcal{A}$) as long as $(\alpha_K, \alpha_K)=1$. Let $g \in G$, $v \in V$. We have

$$\begin{aligned} f(g(v)) &= \sum_{K \in \mathcal{A}} (g(v), \alpha_K)\alpha_K = \sum_{K \in \mathcal{A}} (v, g^{-1}(\alpha_K))\alpha_K \\ &= g[\sum_{K \in \mathcal{A}} (v, g^{-1}(\alpha_K))g^{-1}(\alpha_K)] = g(f(v)). \end{aligned}$$

By Schur's lemma, there exists a constant number C such that $f(v)=Cv$ for all $v \in V$. Let $e_1, e_2, \cdots, e_l$ be an orthonormal basis of V. To determine C we compute

$$Cl=\sum_{i=1}^{l} (f(e_i), e_i)=\sum_{i=1}^{l} \sum_{K \in \mathcal{A}} |(e_i, \alpha_K)|^2 = \sum_{K \in \mathcal{A}} |\alpha_K|^2 = n.$$

Thus $C=n/l=t/2$.

Proposition 2.2. *Let $H \in \mathcal{A}$. Then*

$$t=2 \sum_{K \in \mathcal{A}} |(\alpha_H, \alpha_K)|^2.$$

Proof. By Proposition 2.1, we have

$$t|v|^2=(tv, v)=2 \sum_{K \in \mathcal{A}} |(v, \alpha_K)|^2.$$

Put $v=\alpha_H$.

Remark. When G is defined over the real number field, Proposition 2.2 is the known formula [1, Ch. 5, §6.2, Th. 1, Cor.].

Fix $H \in \mathcal{A}$. Let n'' denote the cardinality of $\mathcal{A}''=\{H \cap K \mid K \in \mathcal{A}, K \neq H\}$.

Theorem 2.3.

$$t=n-n''+1.$$

In particular the cardinality of $\mathcal{A}''$ does not depend on $H \in \mathcal{A}$.

Corollary 2.4. *For an irreducible unitary reflection group G, t is an integer.*

The rest is devoted to the proof of Theorem 2.3. Fix $H \in \mathcal{A}$. Define

$$\mathcal{A}''=\{H \cap K \mid K \in \mathcal{A}, K \neq H\}.$$

Then $\mathcal{A}''$ is a finite collection of $(l-2)$-dimensional vector subspaces of H. Let $X \in \mathcal{A}''$. Denote by $X^\perp$ the orthogonal complement of X in V. Then $X^\perp$ is of dimension two. Define

$$\mathcal{A}_X=\{H \in \mathcal{A} \mid X \subset H\}, \qquad \mathcal{A}_X^\perp=\{X^\perp \cap K \mid K \in \mathcal{A}_X\}.$$

Then $\mathcal{A}_X^\perp$ is a finite collection of one-dimensional vector subspaces of $X^\perp$. Let

$$G_X=\{g \in G \mid X \subset \mathrm{Fix}(g)\}.$$

Then G_X is a unitary reflection group by [5, 1.5]. Since $X^\perp$ is G_X-stable, there is a natural injection $G_X \to \mathrm{GL}(X^\perp)$. Denote the image of this injection by $G_X^\perp$. Then $G_X^\perp$ is a unitary reflection group, and the arrangement defined by $G_X^\perp$ is $\mathcal{A}_X^\perp$.

Lemma 2.5.

$$\sharp\mathcal{A}_X = 2\sum_{K\in\mathcal{A}_X} |(\alpha_H, \alpha_K)|^2.$$

Proof. Note that $\alpha_K \in X^\perp$ for $K \in \mathcal{A}_X$.

Case 1. If $G_X^\perp$ is irreducible, we can apply Proposition 2.2 to get the desired result.

Case 2. If $G_X^\perp$ is reducible, then

$$\sharp\mathcal{A}_X = \sharp\mathcal{A}_X^\perp = 2.$$

Let $\mathcal{A}_X = \{H, K\}$. Then $(\alpha_H, \alpha_K) = 0$. So, in this case,

$$\sharp\mathcal{A}_X = 2 = 2\sum_{K\in\mathcal{A}_X} |(\alpha_H, \alpha_K)|^2.$$

Define

$$\mathcal{A}' = \mathcal{A}\backslash\{H\}, \quad \mathcal{A}'_X = \mathcal{A}_X\backslash\{H\} \quad (X \in \mathcal{A}'').$$

Then the following lemma is obvious:

Lemma 2.6.

$$\mathcal{A}' = \bigcup_{X\in\mathcal{A}''} \mathcal{A}'_X \quad (\textit{disjoint}).$$

We give a proof of Theorem 2.3 by applying Proposition 2.2, Lemma 2.6, and Lemma 2.5:

$$\begin{aligned}
t &= 2\sum_{K\in\mathcal{A}} |(\alpha_H, \alpha_K)|^2 = 2 + 2\sum_{K\in\mathcal{A}'} |(\alpha_H, \alpha_K)|^2 \\
&= 2 + 2\sum_{X\in\mathcal{A}''}\sum_{K\in\mathcal{A}'_X} |(\alpha_H, \alpha_K)|^2 \\
&= 2 + \sum_{X\in\mathcal{A}''}\Big(2\sum_{K\in\mathcal{A}_X} |(\alpha_H, \alpha_K)|^2 - 2\Big) \\
&= 2 + \sum_{X\in\mathcal{A}''}(\sharp\mathcal{A}_X - 2) = 2 + \sum_{X\in\mathcal{A}''}(\sharp\mathcal{A}_X - 1) - \sharp\mathcal{A}'' \\
&= 2 + \sum_{X\in\mathcal{A}''}\sharp\mathcal{A}'_X - n'' = 2 + \sharp\mathcal{A}' - n'' = 1 + n - n''.
\end{aligned}$$

References

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