61. On Siegel Series for Hermitian Forms. II

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This is a continuation of our paper [4]. In that paper we studied the Siegel series $b(s, H)$ for Hermitian form H. Here we shall give some applications of our previous result.

Let $\boldsymbol{K}$ be an imaginary quadratic number field with ring of integers $\mathfrak{o}_{\boldsymbol{K}}$. Let $\boldsymbol{H}_n$ be the Hermitian upper-half space:

$$\boldsymbol{H}_n=\{Z \in M_n(\boldsymbol{C}) \mid (2i)^{-1}(Z-{}^t\bar{Z})>0\}$$

where ${}^t\bar{Z}$ is the transpose complex conjugate to Z. Put

$$J_n=\begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

The Hermitian symplectic group of degree n is defined as

$$\Omega_n=\{M \in M_{2n}(\boldsymbol{C}) \mid {}^t\bar{M}J_nM=J_n\}.$$

The group Ω_n operates on $\boldsymbol{H}_n$ by the action

$$M: Z \longmapsto (AZ+B)(CZ+D)^{-1}, \qquad M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Omega_n.$$

The Hermitian modular group of degree n associated with $\boldsymbol{K}$ is defined as

$$\Gamma_n(\mathfrak{o}_{\boldsymbol{K}})=\Omega_n \cap M_{2n}(\mathfrak{o}_{\boldsymbol{K}}).$$

We denote by $A_k(\Gamma_n(\mathfrak{o}_{\boldsymbol{K}}))$ the complex vector space of Hermitian modular forms for $\Gamma_n(\mathfrak{o}_{\boldsymbol{K}})$ of weight k.

For a rational integer k, we define a function $E_k^{(n)}(Z, s)$ on $\boldsymbol{H}_n \times \boldsymbol{C}$ by

$$E_k^{(n)}(Z, s)=\sum |CZ+D|^{-k} \|CZ+D\|^{-s}, \qquad (Z, s) \in \boldsymbol{H}_n \times \boldsymbol{C},$$

where the sum extends over the representatives of the classes of coprime Hermitian pairs $\{C, D\}$. We know that the series is absolutely convergent if $\mathrm{Re}(s)>2n-k$.

In the rest of this paper, we shall consider the case $n=2$, $\boldsymbol{K}=\boldsymbol{Q}(i)$ and $\mathfrak{o}_{\boldsymbol{K}}=\boldsymbol{Z}[i]$.

Let $k \geq 4$ be a rational integer such that $k \equiv 0 \bmod 4$. We put

$$\psi_k(Z)=\lim_{s \to 0} E_k^{(2)}(Z, s).$$

Then $\psi_k(Z)$ is holomorphic on $\boldsymbol{H}_2$. Furthermore $\psi_k(Z)$ is contained in $A_k(\Gamma_2(\mathfrak{o}_{\boldsymbol{K}}))$, and is called the *Eisenstein series* for $\Gamma_2(\mathfrak{o}_{\boldsymbol{K}})$ of weight k. Here it should be noted that the holomorphy in the case $k=4$ is a consequence of the general result of Shimura [6].

Let

$$\psi_k(Z)=\sum_{0 \leq H \in \Lambda_2(\boldsymbol{K})} a_k(H) \exp[2\pi i tr(HZ)]$$

be the Fourier expansion of $\psi_k(Z)$ where $\Lambda_2(\boldsymbol{K})$ is the set of semi-integral Hermitian matrices for $\boldsymbol{K}$ of degree 2 (cf. [4]).

Our first result is as follows:

Theorem 1. *If $H \in \Lambda_2(K)$ is positive definite, then the Fourier coefficient $a_k(H)$ is given by*

$$a_k(H)=2^2\rho_k d(H)^{k-2}b(k, H)$$

where $\rho_k=\pi^{2k-1}\{(k-2)!\,(k-1)!\}^{-1}$, $d(H)=|2iH|$ *and* $b(s, H)$ *is the Siegel series for* H (cf. [4]).

Example 1. From [4], we have $b(s, H)=\zeta(s)^{-1}L(s-1;\chi)^{-1}F(s, H)$ if $H>0$, where $F(s, H)$ is a function defined in [4] and its value can be computed from Theorem 1 in [4].

$$b\left(4, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)=2700\pi^{-7}, \qquad b\left(4, \begin{pmatrix}1 & 1/2\\ 1/2 & 1\end{pmatrix}\right)=2560\pi^{-7}.$$

Remark 1. For a matrix $H \in \Lambda_2(K)$ of rank<2, $a_k(H)$ is obtained as the Fourier coefficient of the normalized Eisenstein series of lower degree and same weight. Namely we have

$$a_k(H)=\begin{cases}(2\pi)^k\{(k-1)!\,\zeta(k)\}^{-1}\sigma_{k-1}(d_1(H)) & \text{if } |H|=0,\ H\neq 0^{(2)},\\ 1 & \text{if } H=0^{(2)},\end{cases}$$

where $\sigma_j(m)=\sum_{0<d|m} d^j$ and $d_1(H)$ was defined in [4].

In connection with the result of Shimura [6], we are interested in the case $k=4$.

Example 2. From Theorem 1, $a_4(H)$ is given as follows:

$$a_4(H)=\begin{cases}960d(H)^2F(4, H) & \text{if } |H|>0,\\ 240\sigma_3(d_1(H)) & \text{if } |H|=0,\ H\neq 0^{(2)},\\ 1 & \text{if } H=0^{(2)}.\end{cases}$$

It follows from [4] that $d(H)^{k-2}F(k, H)$ is rational integral. Hence $a_4(H)$ is rational integral for any H.

$$a_4\left(\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}\right)=2160, \qquad a_4\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)=14400, \qquad a_4\left(\begin{pmatrix}1 & 1/2\\ 1/2 & 1\end{pmatrix}\right)=7680,$$

$$a_4\left(\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right)=240, \qquad a_4\left(\begin{pmatrix}1 & (1+i)/2\\ (1-i)/2 & 1\end{pmatrix}\right)=2880.$$

Let S denote a positive Hermitian form of rank k. If $\mathcal{L}$ is a lattice in $M_{k\times 2}(C)$, then the theta nullwerthe of order S is

$$\Theta(Z, S)=\sum_{X\in\mathcal{L}} \exp\,[\pi i tr(S[X]Z)], \qquad Z \in H_2,$$

where $S[X]={}^t\overline{X}SX$. If, in particular, $\mathcal{L}=M_{k\times 2}(\mathfrak{o}_K)$, S is even integral over $\mathfrak{o}_K=Z[i]$ and $|S|=1$, then $\Theta(Z, S) \in A_k(\Gamma_2(\mathfrak{o}_K))$. Furthermore, the Fourier expansion is given as

$$\Theta(Z, S)=\sum A(S, H) \exp\,[2\pi i tr(HZ)],$$
$$A(S, H)=\#\{X \in M_{k\times 2}(\mathfrak{o}_K) \mid S[X]=2H\}.$$

Iyanaga's form

$$I=\begin{pmatrix}2 & -i & -i & 1\\ i & 2 & 1 & i\\ i & 1 & 2 & 1\\ 1 & -i & 1 & 2\end{pmatrix}$$

is a representative of the unique class of unimodular positive Hermitian forms in 4 variables which are even integral over $\mathfrak{o}_K=Z[i]$ (cf. [1], [3]). Therefore we have $\Theta(Z, I) \in A_4(\Gamma_2(Z[i]))$. On the other hand, E. Freitag [2] constructed the 6 generators of the graded ring $A(\Gamma_2(Z[i]))=\oplus_{k=0}^{\infty} A_k(\Gamma_2(Z[i]))$

and showed that $A_4(\Gamma_2(\boldsymbol{Z}[i]))$ is one-dimensional, namely,

$$A_4(\Gamma_2(\boldsymbol{Z}[i]))=\boldsymbol{C}\varphi_4(Z),$$

where $\varphi_4(Z)$ is a Hermitian modular form of weight 4 constructed by means of theta functions with characteristic (cf. [2] and Theorem 4 in [1]). By comparing the constant terms of the Fourier expansions of $\psi_4(Z)$, $\Theta(Z, I)$ and $\varphi_4(Z)$, we have the following result.

Theorem 2. *We have*

$$\psi_4(Z)=\Theta(Z, I)=4^{-1}\varphi_4(Z).$$

In particular,

$$A(I, H)=\sharp\{X \in M_{4\times 2}(\boldsymbol{Z}[i]) | I[X]=2H\}=a_4(H).$$

Remark 2. Theorem 2 is a Hermitian version of Raghavan's result [5].

References

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