3. *On the Cauchy-Kowalewski Theorem for Characteristic Initial Surfaces*

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Introduction. We consider the Cauchy problem in the category of holomorphic functions. When the initial surface is non-characteristic, of course we have the well-known theorem of Cauchy-Kowalewski. On the other hand, when it is simply characteristic, Cauchy problem with m initial data is not soluble and that with $m-1$ initial data is not unique (m is the order of the equation), see [4]. Our aim is to show, when the initial surface is characteristic and the multiplicity varies, Cauchy problem with $m-1$ initial data can be uniquely soluble; we give sufficient conditions. The Fuchs type operator with weight $m-1$ will be a particular case.

1. Problem. Let U be a neighborhood of the origin in C^{n+1},

$$P(t,x;\partial_t,\partial_x)=\sum_{s=0}^{m}\sum_{|\alpha|\leqslant s}a_{m-s,\alpha}(t,x)\partial_t^{m-s}\partial_x^\alpha \tag{1}$$
$$a_{m-s,\alpha}(t,x)\in\mathcal{O}(U),$$

where $t\in C$, $x=(x_1,\cdots,x_n)\in C^n$, $m\in N$, $\alpha=(\alpha_1,\cdots,\alpha_n)$ multi-index, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $\partial_x^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$, $\partial_t=\partial/\partial t$, $a(t,x)\in\mathcal{O}(U)$ implies that $a(t,x)$ is defined and holomorphic in U and so is $b(x)\in\mathcal{O}(U_0)$, $U_0=U\cap\{t=0\}$. We denote by P_m the principal part of P and $P_{m(q,\beta)}^{(p,\alpha)}(t,x;\tau,\xi)=\partial_\tau^p\partial_\xi^\alpha\partial_t^q\partial_x^\beta P_m(t,x;\tau,\xi)$, $\tau\in C$, $\xi=(\xi_1,\cdots,\xi_n)\in C^n$.

Assumption A. The hyperplane $t=0$ is characteristic for the operator $P(t,x;\partial_t,\partial_x)$ but not simply characteristic, i.e.

$$P_m(0,x;\tau,0)\equiv 0,\quad P_m^{(0,\alpha)}(0,0;\tau,0)=0\qquad\text{for all } |\alpha|=1. \tag{2}$$

Under the assumption A, we consider the Cauchy problem

$$(P,m-1):\begin{cases}P(t,x;\partial_t,\partial_x)u=f(t,x)\in\mathcal{O}(U)\\ \partial_t^k u|_{t=0}=g_k(x)\in\mathcal{O}(U_0), & k=0,1,\cdots,m-2.\end{cases}$$

When there is a neighborhood of the origin V and a unique solution $u\in\mathcal{O}(V)$, we say simply that the Cauchy problem $(P,m-1)$ is uniquely soluble in $\mathcal{O}$.

2. Characteristic coefficients. To state the results, we need to introduce some quantities. First, let

$$\lambda_0=(\partial_t a_{m,0})(0,0),\qquad \mu=a_{m-1,0}(0,0). \tag{3}$$

Next, we consider the matrix

$$\left((\partial a_{m-1,e_i}/\partial x_j)(0,0);\ \begin{matrix}i:1\downarrow n\\ j:1\rightarrow n\end{matrix}\right) \tag{4}$$

where e_i is the n-dimensional i-th unit vector. Let $\lambda_1,\cdots,\lambda_n$ be the eigenvalues of this matrix. In this paper, we call $\{\lambda_0,\lambda_1,\cdots,\lambda_n,\mu\}$ characteristic coefficients. We may suppose for some k $(0\leqslant k\leqslant n)$

(5) $$\lambda_1, \cdots, \lambda_k \neq 0, \qquad \lambda_{k+1}=\cdots=\lambda_n=0.$$

3. Results. When $\lambda_0\neq 0$, we put the following three conditions.

Condition 1. $p\lambda_0+\beta_1\lambda_1+\cdots+\beta_k\lambda_k+\mu\neq 0$ for every p, $\beta_i\in N\cup\{0\}$.

Condition 2. If we denote by Λ the convex hull of $\{\lambda_0, \lambda_1, \cdots, \lambda_k\}$ on the complex number plane, then $0\notin\Lambda$.

Condition 3. For every $|\alpha|=1$,

(6) $$P_m^{(0,\alpha)}(0,0,x'';\tau,0,0)\equiv 0$$

where $x=(x',x'')$, $x'=(x_1,\cdots,x_k)$, $x''=(x_{k+1},\cdots,x_n)$ and so is $\xi=(\xi',\xi'')$.

Theorem 1. *Assume the assumption* A, $\lambda_0\neq 0$ *and three Conditions* 1, 2 *and* 3, *then the Cauchy problem* $(P, m-1)$ *is uniquely soluble in* $\mathcal{O}$.

When $\lambda_0=0$, we put the following four conditions.

Condition 0*. $k\geq 1$.

Condition 1*. $\beta_1\lambda_1+\cdots+\beta_k\lambda_k+\mu\neq 0$, for every $\beta_i\in N\cup\{0\}$.

Condition 2*. If we denote by Λ^* the convex hull of $\{\lambda_1,\cdots,\lambda_k\}$ on the complex number plane, then $0\notin\Lambda^*$.

Condition 3*. There is an integer h $(0\leqslant h\leqslant k)$ such that, if we denote $x=(x',x'',x''')$, $x'=(x_1,\cdots,x_h)$, $x''=(x_{h+1},\cdots,x_k)$, $x'''=(x_{k+1},\cdots,x_n)$ and $\xi=(\xi',\xi'',\xi''')$, $\alpha=(\alpha',\alpha'',\alpha''')$, $\beta=(\beta',\beta'',\beta''')$ in the same way, then for every $|\alpha'|+|\beta'|\leqslant 1$

(7) $$P_{m(0,\beta',0,0)}^{(0,\alpha',0,0)}(t,0,0,x''';\tau,0,\xi'',\xi''')\equiv 0.$$

Theorem 2. *Assume the assumption* A, $\lambda_0=0$ *and four Conditions* 0*, 1*, 2* *and* 3*, *then the Cauchy problem* $(P,m-1)$ *is uniquely soluble in* $\mathcal{O}$.

4. Outline of the proof. We write

$$a_{m-s,\alpha}(t,x)=\sum_{p=0}^{\infty} a_{m-s,\alpha;p}(x)t^p/p!$$

$$f(t,x)=\sum_{p=0}^{\infty} f_p(x)t^p/p!$$

$$u(t,x)=\sum_{p=0}^{\infty} u_p(x)t^p/p!.$$

We denote

$$L_{p,r}=\sum_{s=0}^{\min\{r,m\}}\sum_{|\alpha|\leqslant s}\binom{p-m}{r-s}a_{m-s,\alpha;r-s}(x)\partial_x^\alpha,$$

$r=0,1,\cdots,p$. Especially $L_{p,0}=a_{m,0;0}(x)\equiv 0$ and

(8) $$L_{p,1}=(p-m)a_{m,0;1}(x)+\sum_{|\alpha|\leqslant 1}a_{m-1,\alpha;0}(x)\partial_x^\alpha.$$

We have then a recurrence relation

(9) $$L_{p+1,1}u_p=f_{p+1-m}-\sum_{r=2}^{p+1}L_{p+1,r}u_{p+1-r}, \qquad p=m-1,m,\cdots.$$

Each u_p will be determined by solving this first order equation. We first study the unique solubility of (9). We should remark that $L_{p+1,1}$ is a first order operator whose principal symbol degenerates at $x=0$. Second, we investigate the convergence of the series $\sum u_p(x)t^p/p!$. For details, see our forthcoming paper.

5. Remarks. a) When $k=0$, Theorem 1 is the result obtained by

Y. Hasegawa, [2]. In that case, the equation is said to be of Fuchs type with weight $m-1$, see [1].

b) When $k=n$, the assumption A includes the condition 3.

c) Concerning the first order equations with degenerate principal symbol, we have many results, see e.g. T. Oshima [5].

d) We give some examples of 2nd order operators for which the Cauchy problem $(P, 1)$ with initial plane $t=0$ is not uniquely soluble in $\mathcal{O}$; there are divergent power series solutions. They don't satisfy the condition 3 or the condition 3*.

Example 1. $P=t\partial_t^2+bx^2\partial_x\partial_t+c\partial_t$,

b, c constants, $b\neq 0$, $c\neq 0, -1, -2, \cdots$.

Example 2. $P=x\partial_x\partial_t+b\partial_y^2+c\partial_t$,

b, c constants, $b\neq 0$, $c\neq 0, -1, -2, \cdots$.

Example 3. $P=atx\partial_t^2+x\partial_x\partial_t+b\partial_x^2+c\partial_t$,

a, b, c constants, $a<0$, $b<0$, $c>0$.

References

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