57. *On Artinian Modules*

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Introduction. Let R be a Noetherian ring, I an ideal of R and A an Artinian R-module. Matlis [3] has defined the notions of A-cosequence and of I-dimension of A, which are characterized by the Koszul complex. Now let R, I be as above but A a finitely generated R-module. The notions of A-sequence and of $\mathrm{depth}_I A$ are well-known in commutative algebra. $\mathrm{Depth}_I A$ is characterized by the Koszul complex or alternatively using the functor Ext.

We see a parallelism in these notions; we find correspondence between A-sequence and A-cosequence and between $\mathrm{depth}_I A$ and I-dimension of A. For this reason, we shall call the latter the *I-codepth* of A and write $\mathrm{codepth}_I A$. We shall show in §1 of this paper that it can be characterized by Ext. We shall show some more properties of codepth in §2, and give some examples in §3.

Throughout this paper, R is a commutative Noetherian ring with 1. If A is an R-module, then $E(A)$ denotes the injective envelope of A. If I is an ideal of R then $V(I)$ denotes the set of prime ideals containing I.

§1. Characterization of codepth by Ext.

Definition. Let R be a Noetherian ring, I an ideal of R, A an Artinian R-module and $x_1, \cdots, x_n$ elements of R. Then a sequence $x_1, \cdots, x_n$ is said to be an *A-cosequence* if

1) $E_i \xrightarrow{x_{i+1}} E_i \longrightarrow 0$ exact $(i=0, 1, \cdots, n-1)$
where $E_0 = A$, $E_i = 0 :_A (x_1, \cdots, x_i)$ if $i \neq 0$.
2) $E_n = 0 :_A (x_1, \cdots, x_n) \neq 0$.

Remark. Let R, I, A be as above. If $x_1, \cdots, x_n$ is an A-cosequence in I, then the ideals (x_1), (x_1, x_2), $\cdots$, $(x_1, x_2, \cdots, x_n)$ form a properly ascending chain. Therefore, every A-cosequence can be extended to a maximal one which has finite length.

Definition. Let R be a Noetherian ring with a proper ideal I. Let A be an Artinian R-module. Then the *I-codepth* of A, $\mathrm{codepth}_I A$ is the length of the longest A-cosequence in I.

If R is a local ring with a maximal ideal M, $\mathrm{codepth}_M A$ is called simply the codepth of A, codepth A.

Theorem 1. *Let A be an Artinian R-module, I an ideal of R with $0 :_A I \neq 0$ and E an injective cogenerator of R. A^* will denote $\mathrm{Hom}_R(A, E)$. Let $n > 0$ be an integer, then the following statements are equivalent.*

1) $\operatorname{Ext}_R^i(N, A^*)=0$ $(i<n)$ *for every finitely generated* R*-module* N *with* $\operatorname{Supp}(N)\subset V(I)$.

2) $\operatorname{Ext}_R^i(R/I, A^*)=0$ $(i<n)$.

3) *There exists an* A*-cosequence of the length* n *in* I.

Proof. 1)$\rightarrow$2). Trivial.

2)$\rightarrow$3). We prove this by induction on n. If $n=1$, we have

$$0=\operatorname{Ext}_R^0(R/I, A^*)=\operatorname{Hom}_R(R/I, A^*)\simeq\operatorname{Hom}_R(A/IA, E).$$

Since E is the injective cogenerator, $A/IA=0$. By Theorem 2 in [3], there exists an element x_1 in I such that $A=x_1A$. Hence x_1 is an A-cosequence in I.

If $n>1$, there exists an A-cosequence x_1 in I. If we put $B=0:{}_Ax_1$, we have the exact sequence:

$$0\longrightarrow B\longrightarrow A\xrightarrow{x_1}A\longrightarrow 0$$

where x_1 above the arrow identifies the map as multiplication by x_1. Since E is injective, the sequence

$$0\longrightarrow A^*\longrightarrow A^*\longrightarrow B^*\longrightarrow 0 \qquad \text{where } B^*=\operatorname{Hom}_R(B, E)$$

is exact. From this, we get the long exact sequence:

$$\cdots\longrightarrow\operatorname{Ext}_R^i(R/I, A^*)\longrightarrow\operatorname{Ext}_R^i(R/I, B^*)\longrightarrow\operatorname{Ext}_R^{i+1}(R/I, A^*)\longrightarrow\cdots$$

which shows that${}_R^i(R/I, B^*)=0$ $(i<n-1)$. By induction hypothesis, there exists a B-cosequence $x_2, \cdots, x_n$ in I. Hence $x_1, \cdots, x_n$ is an A-cosequence.

3)$\rightarrow$1). We prove this again by induction on n. If $n=1$, we have

$$\operatorname{Ext}_R^0(N, A^*)\simeq\operatorname{Hom}_R(N\otimes_R A, E).$$

Since $\operatorname{Supp}(N)\subset V(I)$, x_1 is contained in the radical of $\operatorname{Ann}_R(N)$. And by $A=x_1A$, we have $N\otimes_R A=0$. Hence $\operatorname{Ext}_R^0(N, A^*)=0$.

Suppose now $n>1$. Since $x_1, x_2, \cdots, x_{n-1}$ is an A-cosequence, we get $\operatorname{Ext}_R^i(N, A^*)=0$ for $(i<n-1)$ by induction hypothesis. Thus we only need $\operatorname{Ext}_R^{n-1}(N, A^*)=0$. Now $x_2, \cdots, x_n$ is a B-cosequence. By induction hypothesis, $\operatorname{Ext}_R^i(N, B^*)=0$ for $i<n-1$. In particular, $\operatorname{Ext}_R^{n-2}(N, B^*)=0$. Thus we get

$$0=\operatorname{Ext}_R^{n-2}(N, B^*)\longrightarrow\operatorname{Ext}_R^{n-1}(N, A^*)\xrightarrow{x_1}\operatorname{Ext}_R^{n-1}(N, A^*) \quad \text{exact.}$$

Thus the multiplication by x_1 is injective. But x_1 is an element of the radical of $\operatorname{Ann}_R(N)$. Therefore there exists an integer m such that x_1^m annihilates N. Hence x_1^m annihilates $\operatorname{Ext}_R^{n-1}(N, A^*)$ as well. Thus we have $\operatorname{Ext}_R^i(N, A^*)=0$ $(i<n)$.

This completes the proof of theorem.

Corollary 2. *Under the same assumption as above, we have*

$$\operatorname{codepth}_I A=\inf\{n\,|\,\operatorname{Ext}_R^n(R/I, A^*)\neq 0\}.$$

Corollary 3. *Let* R, E *be as above and* $0\rightarrow A\rightarrow B\rightarrow C\rightarrow 0$ *an exact sequence of Artinian* R*-modules. Let* I *be an ideal of* R *with* $0:{}_AI\neq 0$ *and* $0:{}_CI\neq 0$.

1) *If* $\operatorname{codepth}_I B<\operatorname{codepth}_I A$, *then* $\operatorname{codepth}_I C=\operatorname{codepth}_I B$.

2) *If* $\operatorname{codepth}_I B>\operatorname{codepth}_I A$, *then* $\operatorname{codepth}_I C=\operatorname{codepth}_I A+1$.

3) *If* $\operatorname{codepth}_I B=\operatorname{codepth}_I A$, *then* $\operatorname{codepth}_I C\geqq\operatorname{codepth}_I A$.

§2. **Other properties of codepth.** In this section, we show some other properties of codepth. Firstly, we show a relation between Krull dimension and codepth.

Proposition 4. *Let R be a local ring and A an Artinian R-module, then the following inequality holds.*

$$\text{codepth } A \leqq \dim A.$$

Proof. Put $n = \text{codepth } A$. We prove the proposition by induction on n. If $n=0$, it is clear. If $n>0$, we have an A-cosequence x. Put $B=0 :_A x$. By induction hypothesis, we have $\text{codepth } B \leqq \dim B$. Since $\text{codepth } A = \text{codepth } B+1$, it suffices to prove $\dim B+1 \leqq \dim A$. For brief, we put $\mathfrak{a} = \text{Ann}_R A$, $\bar{R}=R/\mathfrak{a}$. Let $\bar{x}$ be an image of x in $\bar{R}$, then we get $\dim B \leqq \dim \bar{R}/(\bar{x})$. Since $\bar{x}$ is a non-zero-divisor in $\bar{R}$, we get $\dim \bar{R}/(\bar{x}) \leqq \dim \bar{R} - 1 = \dim A - 1$. Q.E.D.

Let R be a local ring and A a finitely generated R-module. We know that $\text{depth } A \leqq \dim A$. A is called Cohen-Macaulay (briefly, CM) if $\text{depth } A = \dim A$ or $A=0$. Now we define a co-CM-module.

Definition. Let R be a local ring. An Artinian R-module A is said to be a *co-CM-module* if $\text{codepth } A = \dim A$ or $A=0$.

Proposition 5. *Let A be a finitely generated R-module and E an injective cogenerator of R. Let $x_1, \cdots, x_n$ be elements of R, then the following statements are equivalent.*

1) $x_1, \cdots, x_n$ is an A-sequence.

2) $x_1, \cdots, x_n$ is an A^*-cosequence where $A^* = \text{Hom}_R(A, E)$.

Proof. We put $I_i = (x_1, \cdots, x_{i-1})$, $I_1 = (0)$.

1)→2). We have the exact sequence

$$0 \longrightarrow A/I_iA \xrightarrow{x_i} A/I_iA, \qquad i=1, \cdots, n.$$

Since E is injective and $0 :_{A^*} I_i$ is isomorphic to $(A/I_iA)^*$ where $(A/I_iA)^* = \text{Hom}_R(A/I_iA, E)$, we get the exact sequence

$$0 :_{A^*} I_i \xrightarrow{x_i} 0 :_{A^*} I_i \longrightarrow 0, \qquad i=1, \cdots, n.$$

Hence $x_1, \cdots, x_n$ is A^*-cosequence.

2)→1). We prove that x_i is a non-zero-divisor on A/I_iA for $i=0, \cdots, n-1$. Otherwise, there exists a non zero element $\bar{a}$ in A/I_iA such that $x_i\bar{a}=0$. Since E is an injective cogenerator, there exists R-homomorphism $\phi : A/I_iA \to E$ such that $\phi(\bar{a}) \neq 0$. Since x_i is $0 :_{A^*} I_i$-cosequence and $(A/I_iA)^* \simeq 0 :_{A^*} I_i$, we get an R-homomorphism $\psi : A/I_iA \to E$ such that $\phi = x_i\Psi$. Hence $\phi(\bar{a}) = x_i\psi(\bar{a}) = 0$. This is contradiction. Hence x_i is a non-zero-divisor on A/I_iA and $x_1, \cdots, x_n$ is an A-sequence.

Corollary 6. *Let R be a local ring with maximal ideal M, A a finitely generated R-module and E an injective envelope of R/M. Then*

$$\text{depth } A = \text{codepth } A^* \quad \textit{where } A^* = \text{Hom}_R(A, E).$$

We conclude this section with an interesting result on a CM-module over a local ring.

Lemma. *Let R be a local ring with a maximal ideal M, A an R-module* *and E an injective envelope of R/M. Write $A^* = \mathrm{Hom}_R(A, E)$, then $0 :_R A = 0 :_R A^*$.*

This is easy to prove, and from this lemma and the preceeding Corollary follows

Theorem 7. *Let R, E be as above and A a finitely generated R-module, then A is a CM-module if and only if A^* is a co-CM-module (where $A^* = \mathrm{Hom}_R(A, E)$).*

§ 3. Examples.

Example 1. Let Z be the ring of integers and Q the Z-module of rational numbers. Let p be a prime number and denote by $Z_{(p)}$ the locarization of Z with respect to (p). We regard $Z_{(p)}$ as Z-module. Then $Q/Z_{(p)}$ is the injective envelope of $Z/(p)$, because $Q/Z_{(p)}$ is divisible and essential over a submodule $\{0/p, 1/p, \cdots, (p-1)/p\}$ which is Z-isomorphic to $Z_{(p)}/(p)Z_{(p)}$. Hence $Q/Z_{(p)}$ is a one dimensional co-CM-Z-module.

Example 2. Let R be a local ring. It is demonstrated in [2] that if A is an Artinian R-module, then the inverse polynomial module $A[X_1^{-1}, \cdots, X_n^{-1}]$ may be given a structure of $R[X_1, \cdots, X_n]$-module. And further $A[X_1^{-1}, \cdots, X_n^{-1}]$ is an Artinian $R[[X_1, \cdots, X_n]]$-module and $R[[X_1, \cdots, X_n]]$ is local. Then codepth $A[X_1^{-1}, \cdots, X_n^{-1}] =$ codepth $A + n$. In particular, if R is a field k, $k[X_1^{-1}, \cdots, X_n^{-1}]$ is a co-CM-$k[[X_1, \cdots, X_n]]$-module.

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