37. *Density of the Range of a Wave Operator with Nonmonotone Superlinear Nonlinearity*

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1. Introduction. In this article we shall study the nonlinear wave equation:

(1) $u_{tt}-u_{xx}+g(u)=f(x,t), \qquad (x,t)\in(0,\pi)\times\boldsymbol{R},$

(2) $u(0,t)=u(\pi,t)=0, \qquad t\in\boldsymbol{R},$

(3) $u(x,t+T)=u(x,t), \qquad (x,t)\in(0,\pi)\times\boldsymbol{R},$

where $T>0$ is a rational multiple of π, $g(s)$ is a continuous function on $\boldsymbol{R}$ and $f(x,t)$ is a given T-periodic function of t.

Many mathematicians concerned with this problem (see [1], [7] and its references). Except for [2, 3, 6, 11, 12] they ask that $g(s)$ is monotonic, in order to overcome the lack of compactness due to the fact that the kernel of the wave operator $\partial_t^2-\partial_x^2$ is infinite dimensional.

Working in a restricted class $\tilde{H}$ of functions satisfying some symmetry properties and such that

(i) $\tilde{H}\cap\mathrm{Ker}\,(\partial_t^2-\partial_x^2)=\{0\}$,

(ii) $\tilde{H}$ is invariant under $\partial_t^2-\partial_x^2$ and g,

J. M. Coron [3] proved the existence of multiple T-periodic solutions of (1)–(3) in case $f\equiv 0$ and the existence of forced vibrations under the condition $f\in\tilde{H}$ without assumption of monotonicity. See also N. Basile and M. Mininni [2].

On the other hand, M. Willem [11, 12] and H. Hofer [6] also dealt with the problem (1)–(3) without the monotonicity assumption. They tackled the infinite dimensional kernel of $\partial_t^2-\partial_x^2$ without introducing restricted classes. Under the following *nonresonance* condition:

(4) For consecutive eigenvalues $\alpha<\beta$ of $-(\partial_t^2-\partial_x^2)$ and for some constants $\varepsilon>0$, $r>0$,

$$\alpha+\varepsilon\leqq\frac{g(s)}{s}\leqq\beta-\varepsilon \qquad \text{for } |s|\geqq r,$$

and some additional conditions, they proved that (1)–(3) is *almost solvable*; (1)–(3) possesses a solution for a *dense set* of f's in L^2, in other words, the range of the operator: $u\to u_{tt}-u_{xx}+g(u)$ is dense in L^2. Their arguments are based on the variational methods; [11, 12] used I. Ekeland's variational principles (c.f. [4]), [6] used Leray-Schauder theory in conjunction with the variational method. Note that under the condition (4) the solutions of (1)–(3) are a priori bounded in L^2. See also K. Tanaka [10].

This paper is an extension of [6, 10, 11, 12] and deals with the case that $g(s)$ is *superlinear* (i.e., $g(s)/s\to\infty$ as $|s|\to\infty$). Furthermore, we need not assume that $\sigma(\partial_t^2-\partial_x^2)=\{j^2-(2\pi/T)^2k^2;\ j\in\boldsymbol{N},\ k\in\boldsymbol{Z}\}$ is a discrete set of $\boldsymbol{R}$. In particular, $T/\pi\in\boldsymbol{Q}$ *is not assumed*. In the arguments of [6, 10, 11, 12] it is essential that $\sigma(\partial_t^2-\partial_x^2)$ is discrete. Our main result is as follows:

Theorem. *Assume that there exist constants C_1, $C_2>0$ such that*

$$G(s)\equiv\int_0^s g(\tau)d\tau\leqq C_1 sg(s)+C_2 \qquad \textit{for all } s\in\boldsymbol{R}, \tag{5}$$

$$\lim_{|s|\to\infty}\frac{g(s)}{s}=\infty. \tag{6}$$

Then, (1)–(3) *is almost solvable*, i.e., (1)–(3) *possesses a solution for a dense set of f's in L^2.*

Remarks. 1. Under the conditions (5), (6), we do not have a priori estimates for solutions of (1)–(3) (c.f. [7, 8, 9]). 2. H. Brézis [1] conjectured that *when $g(u)=u^3$, problem* (1)–(3) *possesses a solution—even infinitely many solutions—for every f (or at least for a dense set of f's)*. Our result shows that the weakest statement of his conjecture is true. Concerning the existence of infinitely many periodic solutions, we refer to K. Tanaka [8, 9].

To prove Theorem, as in [10] we shall approximate the wave equation (1)–(3) by the nonlinear telegraph equations:

$$u_{tt}-u_{xx}+\varepsilon u_t+g(u)=f(x,t), \qquad (x,t)\in(0,\pi)\times\boldsymbol{R}, \tag{7}$$

$$u(0,t)=u(\pi,t)=0, \qquad t\in\boldsymbol{R}, \tag{8}$$

$$u(x,t+T)=u(x,t), \qquad (x,t)\in(0,\pi)\times\boldsymbol{R}, \tag{9}$$

for $\varepsilon>0$. Leray-Schauder theory ensures the existence of a solution u^ε of (7)–(9). We shall prove Theorem by showing $\|\varepsilon u_t^\varepsilon\|\to 0$.

2. Proof of Theorem. Let C^∞ be the real vector space of arbitrarily often continuously differentiable functions in $(0,\pi)\times\boldsymbol{R}$, which are T-periodic in t and satisfy $u(0,t)=u(\pi,t)=0$ for all t. We denote by L^2 the completion of C^∞ with respect to the norm $\|u\|=(u,u)^{1/2}$, where $(u,v)=\iint_\Omega uv\,dx\,dt$, $\Omega=(0,\pi)\times(0,T)$. Furthermore, let X be the completion of C^∞ with respect to the norm

$$\|u\|_X=\sup_{t\in[0,T]}\left\{\int_0^\pi(|u_x(x,t)|^2+|u_t(x,t)|^2)dx\right\}^{1/2},$$

that is, X is the space of T-periodic functions which belong to $C(\boldsymbol{R};H_0^1(0,\pi))\cap C^1(\boldsymbol{R};L^2(0,\pi))$.

A function $u\in X$ is said to be a *weak solution* of (7)–(9) if and only if $(u,\phi_{tt}-\phi_{xx}-\varepsilon\phi_t)+(g(u),\phi)=(f,\phi)$ for all $\phi\in C^\infty$. The following proposition is a special case of A. Haraux [5].

Proposition. *Assume that $\varepsilon>0$, $f\in L^2$ and $g(s)$ satisfies for some constants $\delta>0$, $C_1>0$, $C_2>0$,*

$$g(s)s\geqq-(1-\delta)s^2-C_1|s| \qquad \textit{for all } s, \tag{10}$$

$$G(s)\leqq C_2(1+s^2+sg(s)) \qquad \textit{for all } s. \tag{11}$$

Then, there exists a weak solution $u^\varepsilon\in X$ of (7)–(9).

Proof of Theorem. Let $f \in C^{\infty}$. Note that (10) and (11) follow from (5) and (6). Therefore there exists a weak solution $u^{\varepsilon} \in X$ of (7)–(9).

We multiply the telegraph equation (7) by u^{ε}, integrate over Ω, then we get

(12) $$\|u_x^{\varepsilon}\|^2+(g(u^{\varepsilon}), u^{\varepsilon})=\|u_t^{\varepsilon}\|^2+(f, u^{\varepsilon}).$$

Note that

$$\begin{aligned}(u_{tt}^{\varepsilon}-&u_{xx}^{\varepsilon}+g(u^{\varepsilon}), u_t^{\varepsilon})\\ &=\int_0^T \frac{d}{dt}\left[\int_0^{\pi}\left(\frac{1}{2}(|u_t^{\varepsilon}|^2+|u_x^{\varepsilon}|^2)+G(u^{\varepsilon})\right)dx\right]dt\\ &=0.\end{aligned}$$

Hence multiplying the telegraph equation (7) by u_t^{ε}, we obtain

(13) $$\varepsilon\|u_t^{\varepsilon}\|^2=(f, u_t^{\varepsilon})=-(f_t, u^{\varepsilon})\leqq\|f_t\|\|u^{\varepsilon}\|.$$

From (12) and (13), we find

(14) $$(g(u^{\varepsilon}), u^{\varepsilon})\leqq\frac{1}{\varepsilon}\|f_t\|\|u^{\varepsilon}\|+\|f\|\|u^{\varepsilon}\|.$$

By the assumption (6), for any $L>1$ there exists a constant $C_L>0$ such that

$$g(s)s\geqq Ls^2-C_L \qquad \text{for all } s.$$

From (14) we get

$$\begin{aligned}L\|u^{\varepsilon}\|^2-C_L&\leqq\frac{1}{\varepsilon}\|f_t\|\|u^{\varepsilon}\|+\|f\|\|u^{\varepsilon}\|\\ &\leqq\frac{1}{2\varepsilon^2}\|f_t\|^2+\frac{1}{2}\|u^{\varepsilon}\|^2+\frac{1}{2}\|f\|^2+\frac{1}{2}\|u^{\varepsilon}\|^2.\end{aligned}$$

Hence

$$\|u^{\varepsilon}\|\leqq(L-1)^{-1/2}\left(\frac{1}{2\varepsilon^2}\|f_t\|^2+\frac{1}{2}\|f\|^2+C_L\right)^{1/2}.$$

By (13), we obtain

$$\begin{aligned}\|\varepsilon u_t^{\varepsilon}\|^2&\leqq\varepsilon\|f_t\|\|u^{\varepsilon}\|\\ &\leqq(L-1)^{-1/2}\left(\frac{1}{2}\|f_t\|^2+\frac{\varepsilon^2}{2}\|f\|^2+\varepsilon^2C_L\right)^{1/2}\|f_t\|.\end{aligned}$$

Passing to the limit, we get

$$\limsup_{\varepsilon\to 0}\|\varepsilon u_t^{\varepsilon}\|^2\leqq\frac{1}{2}(L-1)^{-1/2}\|f_t\|^2.$$

Since we can choose $L>1$ arbitrarily large, we obtain

$$\varepsilon u_t^{\varepsilon}\to 0 \qquad \text{strongly in } L^2 \text{ as } \varepsilon\to 0,$$

i.e.,

$$u_{tt}^{\varepsilon}-u_{xx}^{\varepsilon}+g(u^{\varepsilon})=-\varepsilon u_t^{\varepsilon}+f\to f \qquad \text{in } L^2 \text{ as } \varepsilon\to 0.$$

Thus any $f \in C^{\infty}$ belongs to the L^2-closure of the range of the operator: $u\to u_{tt}-u_{xx}+g(u)$. Since C^{∞} is dence in L^2, the proof is completed.

Remark. In the above argument, any property of $\sigma(\partial_t^2-\partial_x^2)$, is not used. So we can get similar result for the equation:

(15) $$u_{tt}-u_{xx}+g(u)=f(x, t), \qquad (x, t)\in D\times \boldsymbol{R},$$

(16) $$u(x, t)=0, \qquad (x, t)\in \partial D\times \boldsymbol{R},$$

(17) $$u(x, t+T)=u(x, t), \qquad (x, t)\in D\times \boldsymbol{R},$$

where $D\subset \boldsymbol{R}^N$ $(N\geqq 2)$ is a bounded domain with a smooth boundary ∂D.

Similarly to the proof of Theorem, we can prove almost solvability of (15)–(17) under assumptions (5), (6) and

(18) $$|g(s)| \leqq C(1+|s|^p) \qquad \text{for all } s \in \boldsymbol{R},$$

for some constants $p<(N/N-2)$ ($p<\infty$ if $N=2$) and $C>0$. Here the condition (18) ensures compactness of the operator : $u \to g(u)$; $X \to L^2$.

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