12. *Random Media and Quasi-Classical Limit of Schrödinger Operator*

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In the present note we consider a mathematical problem concerning random media. We consider a bounded domain Ω in $\boldsymbol{R}^3$ with smooth boundary Γ. We put $B(\varepsilon; w)=\{x\in \boldsymbol{R}^3; |x-w|<\varepsilon\}$. Fix $\beta\geq 1$. Let $0<\mu_1(\varepsilon; w(m))\leq \mu_2(\varepsilon; w(m))\leq \cdots$ be the eigenvalues of $-\Delta$ $(=-\operatorname{div}\operatorname{grad})$ in $\Omega_{\varepsilon, w(m)}=\Omega\backslash\overline{\bigcup_{i=1}^{\tilde{m}} B(\varepsilon; w_i^{(m)})}$ under the Dirichlet condition on its boundary. Here $\tilde{m}$ denotes the largest integer which does not exceed m^β, and $w(m)$ denotes the set of $\tilde{m}$-points $\{w_i^{(m)}\}_{i=1}^{\tilde{m}}\in\Omega^{\tilde{m}}$. Let $V(x)>0$ be C^1-class function on $\overline{\Omega}$ satisfying

$$\int_\Omega V(x)dx=1.$$

We consider Ω as the probability space with the probability density $V(x)dx$. Let $\Omega^{\tilde{m}}=\prod_{i=1}^{\tilde{m}}\Omega$ be the probability space with the product measure. The following result which is an elaboration of M. Kac's theorem (Kac [3]) was given in Ozawa [4].

Theorem A. *Assume that* $\beta=1$. *Fix* $\alpha>0$ *and* k. *Then,*

$$\lim_{m\to\infty} \boldsymbol{P}(w(m)\in\Omega^{\tilde{m}}; m^\delta|\mu_k(\alpha/m; w(m))-\mu_k^V|<\varepsilon)=1$$

holds for any $\varepsilon>0$ *and* $\delta\in[0,1/4)$. *Here* μ_k^V *denotes the* k^{th} *eigenvalue of* $-\Delta+4\pi\alpha V(x)$ *in* Ω *under the Dirichlet condition on* Γ.

In this paper we study the case $\beta>1$. In this case the sum of the radii of $\tilde{m}$-balls $B(\alpha/m; w_i^{(m)})$, $i=1,\cdots,\tilde{m}$, tends to ∞ as $m\to\infty$. We see by the argument in Rauch-Taylor [9] that $\mu_k(\alpha/m; w(m))\to\infty$ if $\beta>1$, $V(x)>0$ and

$$\lim_{m\to\infty}\tilde{m}^{-1}\sum_{i=1}^{\tilde{m}} f(w_i^{(m)})=\int_\Omega f(x)V(x)dx$$

for any fixed $f\in L^\infty(\Omega)$. We call the case $\beta>1$, $V(x)>0$ to be the solidifying case following Rauch-Taylor.

The aim of this paper is to give the following:

Theorem 1. *Assume that* $\beta\in[1,9/8)$ *and* $V(x)>0$. *Fix* $\alpha>0$ *and* k. *Then, there exists a constant* $\delta(\beta)>0$ *independent of* m *such that*

$$\lim_{m\to\infty} \boldsymbol{P}(w(m)\in\Omega^{\tilde{m}}; m^{\delta'-(\beta-1)}|\mu_k(\alpha/m; w(m))-\mu_{k,m}^V|<\varepsilon)=1$$

holds for any $\varepsilon>0$ *and* $\delta'\in[0,\delta(\beta))$. *Here* $\mu_{k,m}^V$ *denotes the* k^{th} *eigenvalue of* $-\Delta+4\pi\alpha\tilde{m}m^{-1}V(x)$ *in* Ω *under the Dirichlet condition on* Γ.

Remark. There exist constants C' and C'' such that $C'<m^{-(\beta-1)}\mu_{k,m}^V<C''$ holds.

Readers may refer to Papanicolaou-Varadhan [7], [8] Simon [10], Bensoussan-Lions-Papanicolaou [1], Huruslov-Marchenko [2], Ozawa [5], [6] and the literatures cited there, for related topics.

We give a sketch of our proof of Theorem 1. Fix $\beta \in (1,3)$. We consider the following condition $(D-0)_m$, $(D-\infty)_m$ on $w(m)$.

$(D-0)_m$: Assume that $\Omega \setminus \overline{\bigcup_{i=1}^{\tilde{m}} B(\alpha/m\,;w_i^{(m)})}$ is divided into the connected components

$$\omega_1(w(m)),\ \cdots,\ \omega_{g(w(m))}(w(m)).$$

Then, $g(w(m))=1$ or

$$\max_{2\leq s\leq g(w(m))} \operatorname{diam} \omega_{s(w(m))}(w(m)) \leq m^{-1}\log m$$

holds. Here diam $\mathcal{Z}$ denotes the diameter of the set $\mathcal{Z}$.

$(D-\infty)_m$: Take an arbitrary connected closed subset $\mathcal{R}_m$ of Γ satisfying diam $\mathcal{R}_m \geq 2m^{-1}\log m$. Then

$$\mathcal{R}_m \setminus \overline{\bigcup_{i=1}^{\tilde{m}} B(\alpha/m\,;w_i^{(m)})} \neq \phi.$$

We can easily get the following:

$$\lim_{m\to\infty} \boldsymbol{P}(w(m) \in \Omega^{\tilde{m}}\,;\, w(m) \text{ satisfies } (D-0)_m,\ (D-\infty)_m)=1.$$

We put $\gamma > \beta-1$. We abbreviate the largest integer which does not exceed m^β as m'. We put $m''=(m')^{1/2}$. Hereafter we always assume that $w(m)$ satisfies $(D-0)_m$, $(D-\infty)_m$. We abbreviate $\omega_1(w(m))$ as ω for the sake of simplicity. Let $G_{(m')}(x,y\,;w(m))$ be the Green's function of $\Delta - m'$ in ω under the Dirichlet condition on its boundary satisfying

$$(\Delta_x - m')G_{(m')}(x,y\,;w(m)) = -\delta(x-y), \qquad x,y \in \omega$$
$$G_{(m')}(x,y\,;w(m))=0, \qquad x \in \partial\omega.$$

Let $G_{(m')}(x,y)$ be the Green's function of $\Delta - m'$ in Ω satisfying

$$(\Delta_x - m')G_{(m')}(x,y) = -\delta(x-y), \qquad x,y \in \Omega$$
$$G_{(m')}(x,y)=0, \qquad x \in \Gamma.$$

From now on we abbreviate $G_{(m')}(x,y)$ as $G(x,y)$. We introduce the following integral kernel function: We abbreviate $w_i^{(m)}$ as w_i for the sake of simplicity.

$$\begin{aligned} h_{(m')}(x,y\,;w(m)) &= G(x,y) - (4\pi\alpha/m)e^{m''\alpha/m} \textstyle\sum_{i=1}^{\tilde{m}} G(x,w_i)G(w_i,y) \\ &\quad + \textstyle\sum_{s=2}^{m^*} (-4\pi\alpha/m)^s e^{m''\alpha s/m} \sum_{(s)} G(x,w_{i_1})G(w_{i_1},w_{i_2}) \\ &\quad \cdots G(w_{i_{s-1}},w_{i_s})G(w_{i_s},y). \end{aligned}$$

Here $m^*=(\log m)^2$ and $m''=(m')^{1/2}$. Here the indices $(i_1,i_2,\cdots,i_s)$ in $\sum_{(s)}$ run over all $1\leq i_1,\cdots,i_s\leq \tilde{m}$ satisfying $i_1 \neq i_2$, $i_2\neq i_3$, $\cdots$, $i_{s-1}\neq i_s$. An essential key to Theorem 1 is the fact that $h_{(m')}(x,y\,;w(m))$, when we consider it as an integral kernel function on $\omega\times\omega$, is a nice approximation of $G_{(m')}(x,y\,;w(m))$ in a rough sense, if $\beta-1$ is small. By a probabilistic consideration we view that $h_{(m')}(x,y\,;w(m))$, when we consider it as an integral kernel function on $\Omega\times\Omega$, is a nice approximation of the integral kernel function of $(-\Delta+m'+4\pi\alpha\tilde{m}m^{-1}V(x))^{-1}$ in a rough sense. Along this line we get Theorem 1. Of course we need hard and long calculations to obtain our result.

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