88. Parametrices and Propagation of Singularities near Gliding Points for Mixed Problems for Symmetric Hyperbolic Systems. II

By Kôji KUBOTA
Department of Mathematics, Hokkaido University

(Communicated by Kôsaku YOSIDA, M. J. A., Nov. 12, 1984)

4. Sketch of proof of Theorem. We follow the procedure in Eskin [2], with some improvements, and modify the construction of the parametrix in [5] which treats the diffractive case where (2) holds with the opposite signature. (The details are given in [13].) We look for the parametrix $E(f)$ in the form:

(6) $$Gv=G_0v_0+G_hv_h+G_ev_e.$$

Here $v={}^t({}^tv_0, {}^tv_h, {}^tv_e)$ is a d^+-vector whose components belong to $H^{-\infty}(R^n)$ and G_h, G_e are operators analogous to the $G^{(2)}, G^{(3)}$ in [5], respectively, while G_0 is an $m\times m_1$ matrix, different essentially from the $G^{(1)}$, whose components are Fourier-Airy integral operators.

To construct G_0 we use such phase functions $\theta(x,\eta')$ and $\rho(x,\eta')$ as in the diffractive case, where $\eta'=(\eta_0,\eta'')\in R^1\times R^{n-1}$. Let $\bar\eta_0=0$ and $\bar\eta''=\bar\xi''$ with $\bar\xi'=(\bar\xi_0,\bar\xi'')$. For definiteness suppose $(\partial\mu/\partial\xi_0)(\bar x,\bar\xi')>0$. Then θ and ρ are real valued functions, defined on a conic neighborhood of $(\bar x,\bar\eta')$, such that $\phi^\pm=\theta\pm(2/3)\rho^{3/2}$ solve the eikonal equation $Q_0(x,\phi_x^\pm)=0$ for $\rho>0$, and that, for $x_n=0$, $\det\theta_{x'\eta'}>0$, $\theta_{x_0\eta_0}>0$ and $\rho_{x_n}<0$ (see [2]). Moreover $\rho(x',0,\eta')=\alpha|\eta'|^{2/3}$, which has been given in [12] and [14], where $\alpha=\eta_0/|\eta'|$, and $Q_0(x,\phi_x^\pm)=O(x_n^\infty)$ as $x_n\to+0$ for $\alpha<0$ and $|\eta'|=1$. Notice that $\theta_{x_n}=\lambda(x,\theta_{x'})$ and $\mu(x,\theta_{x'})=\alpha(\rho_{x_n})^2$ for $x_n=0$ and $|\eta'|=1$. Let $Ai(z)$ be the Airy function of the first kind and set $A_\pm(z)=e^{\mp i\pi/3}Ai(e^{\mp i\pi/3}z)$, which appear in the diffractive case. We then use, as in [2], the Airy function $A_0(z)=A_+(z)+A_-(z)$. It is known that $Ai(z)$ solves $Ai''(z)=zAi(z)$, is an entire function, real valued for real z, and has its zeros only on the negative real axis. Besides, $Ai(0)>0$, $Ai'(0)<0$ and $Ai(z)+\omega Ai(\omega z)+\omega^2Ai(\omega^2z)=0$, where $\omega=e^{i(2/3)\pi}$. Furthermore, for $|z|\gg1$ and $-\pi<\arg z<\pi$, $Ai(z)=z^{-1/4}e^{-(2/3)z^{3/2}}\Psi(z)$ and $\Psi(z)\sim\sum_{k=0}^\infty a_kz^{-(3/2)k}$, where a_k are real and $a_0=(2\sqrt{\pi})^{-1}$. Therefore we have $A_0(z)=Ai(-z)$, $A_\pm(z)=z^{-1/4}e^{\pm i(2/3)z^{3/2}}\Psi_\pm(z)$ and $\Psi_\pm(z)\sim e^{\mp i\pi/4}\cdot\sum_{k=0}^\infty(\pm i)^ka_kz^{-(3/2)k}$ for $|z|\gg1$ and $-\pi\pm\pi/3<\arg z<\pi\pm\pi/3$.

Now let ϕ_1 be the canonical transformation defined by $y'=\theta_{\eta'}(x',0,\eta')$, $\xi'=\theta_{x'}(x',0,\eta')$ and $\phi_1(y',\eta')=(x',\xi')$. Then, under the inverse ϕ_1^{-1} of ϕ_1, the gliding ray $\Gamma(\bar x',\bar\xi')$ is exactly (and locally) mapped onto the straight line, through $(\bar{y}', \bar{\eta}')=\phi_1^{-1}(\bar{x}', \bar{\xi}')$, which is parallel to the y_0 axis and on which y_0 increases as x_0 does. Hereafter we write $y'=(y_0, y'')\in R^1\times R^{n-1}$. Bearing this in mind, we seek G_0 in the form

$$G_0 v_0 = G_1 q_1 v_0 + G_2 q_2 v_0. \tag{7}$$

Here $q_1(y_0)$, $q_2(y_0)$ are cutoff functions such that $q_1+q_2=1$ and $R_0(x', \xi')\neq 0$ on $N_0\cap\phi_1$ (supp q_2). In fact, when (3) is satisfied, we take $q_1=0$, while if this is violated then $G_2 q_2 v_0$ is an additional term, needed only to assure that $v_0(y')\in H^\infty(R^n\cap\{y_0\ll\bar{y}_0\})$. Moreover, for $j=1, 2$, G_j are of the form:

$$(G_j w)(x)=\int e^{i\check{\theta}}(A_0(\check{\rho})\check{a}_j - iA_0'(\check{\rho})\check{b}_j)(A_+(\zeta)^{-1}\chi_1 + A_0(\zeta)^{-1}(1-\chi_1))\hat{w}(\eta')d\eta'. \tag{8}$$

Here

$$\hat{w}(\eta')=\int e^{-iy'\eta'}w(y')dy', \qquad \zeta=(\eta_0-i\tau)|\eta'|^{-1/3},$$

τ being a positive number which is taken large enough, $\check{\rho}(x, \eta')$ is an almost analytic continuation of $\rho(x, \eta')$ with respect to α such that $\check{\rho}(x', 0, \eta')=\zeta$ for $|\eta'|\gg 1$, and $\check{\theta}$, $\check{a}_j$ and $\check{b}_j$ are also defined analogously. (See [2].) The $a_j(x, \eta')$, $b_j(x, \eta')$ are smooth $m\times m_1$ matrices, defined on a conic neighborhood of $(\bar{x}, \bar{\eta}')$, which have the asymptotic expansions $a_j\sim\sum_{k=0}^{-\infty} a_{jk}$, $b_j\sim\sum_{k=0}^{-\infty} b_{jk}$. Here $a_{jk}(x, \eta')$, $b_{jk}(x, \eta')$ are homogeneous in η' of degree k, $k-1/3$, respectively, and if we write

$$e^{-i\theta}P(x, D)\{e^{i\theta}(A_0(\rho)a_j - iA_0'(\rho)b_j)\} = A_0(\rho)c_j - iA_0'(\rho)d_j,$$

$c_j\sim\sum_{k=1}^{-\infty} c_{jk}$, $d_j\sim\sum_{k=1}^{-\infty} d_{jk}$, where $c_{jk}(x, \eta')$, $d_{jk}(x, \eta')$ are homogeneous in η' of degree k, $k-1/3$, respectively, then $c_{jk}=0$ for $\rho\geqq 0$, $c_{jk}=O(x_n^\infty)$ as $x_n\to +0$ for $\alpha<0$, $|\eta'|=1$, and so are d_{jk}. Such a_j, b_j have been constructed in [5], §§3 and 4, so that $a_{j0}=W_1 g_{j0}+\rho W_2 h_{j0}$, $b_{j0}=W_1 h_{j0}+W_2 g_{j0}$, where we have set $W_0(x, \theta_x\pm\sqrt{\rho}\rho_x)=W_1(x, \eta')\pm\sqrt{\rho}W_2(x, \eta')$ and $g_{j0}(x, \eta')$, $h_{j0}(x, \eta')$ are $m_1\times m_1$ matrices homogeneous in η' of degree 0, $-1/3$, respectively. Moreover $\chi_1(\eta')$ is a cutoff function such that $A_0(\alpha|\eta'|^{2/3})\neq 0$ on supp $1-\chi_1$. More precisely, let $\chi(t)\in C^\infty(R^1)$ be a function, supported in $t>3/2$, such that $\chi(t)=1$ for $t>2$ and $\chi'(t)\geqq 0$. Let t_0 be a positive number such that $A_0(t)>0$ for $t\leqq 3t_0$. We then set $\chi_1(\eta')=\chi(\alpha|\eta'|^{2/3}/t_0)$. It should be pointed out that another cutoff function $\chi_\varepsilon(\eta')=\chi(\alpha|\eta'|^\varepsilon)$ with $0<\varepsilon<1/2$ is adopted in [2] and that $(A_0'/A_0)(\zeta)(1-\chi_\varepsilon(\eta'))$ belongs only to a bad class $S_{0,0}^{2/3}$.

In what follows we consider only the more difficult case where (3) is violated and concentrate our attention on the equation $BGv|_{x_n=0}=f$. Noting that $(\check{\theta}-\theta)(x', 0, \eta')\in S_{1,0}^0$, we denote by Φ_1 the Fourier integral operator with phase function $\theta(x', 0, \eta')-y'\eta'$ and with amplitude $e^{i(\check{\theta}-\theta)(x', 0, \eta')}$. Let Φ_1^{-1} be an elliptic Fourier integral operator with the canonical transformation ϕ_1^{-1} such that $\Phi_1\Phi_1^{-1}$ and $\Phi_1^{-1}\Phi_1$ are the identities mod $OPS_{1,0}^{-\infty}$. Suppose $x_n=0$ and $(x', \xi')=\phi_1(y', \eta')$. We then have

(9) $\qquad \Phi_1^{-1}G_j=\tilde{a}_j(1+L\chi_1)+\tilde{b}_j\mathcal{L}, \qquad j=1,2,$

where $\tilde{a}_j, \tilde{b}_j \in OPS^0_{1,0}$ and $\tilde{a}_j(y',\eta')=a_{j0}(x,\eta')$, $\tilde{b}_j(y',\eta')=|\eta'|^{1/3}b_{j0}(x,\eta')$ mod $S^{-1}_{1,0}$. Moreover L, $\mathcal{L}$ are the following Fourier multipliers defined by $(\hat{Lw})(\eta')=L(\eta')\hat{w}(\eta')$ and so on, where $L(\eta')=(A_-/A_+)(\zeta)$, $\mathcal{L}=(K_++K_-L)\chi_1+K_0(1-\chi_1)$, $K_\pm(\eta')=-i|\eta'|^{-1/3}(A'_\pm/A_\pm)(\zeta)$ and $K_0(\eta')=-i|\eta'|^{-1/3}(A'_0/A_0)(\zeta)$. To derive precise estimates for these we set $\gamma=(\alpha^2+|\eta'|^{-4/3})^{1/4}$ and denote constants independent of τ by C and so on. Suppose $|\eta'|\geqq 1$. We then have

$$|\partial^k_{\eta_0}\partial^\beta_{\eta''}K_-(\eta')|\leqq C_{k,\beta}|\eta'|^{-k-|\beta|}\gamma^{1-2k}(1+O(|\eta'|^{-1/3})).$$

The analogous estimates also hold for K_+ and K_0 if $\alpha>0$ and $\alpha|\eta'|^{2/3}\leqq 3t_0$, respectively. In particular, K_-, $K_+\chi_1$ and $K_0(1-\chi_1)$ belong to $S^0_{1/3,0}$. We have also

$$|\partial^k_{\eta_0}\partial^\beta_{\eta''}L(\eta')|\leqq C_{k,\beta}\gamma^{k+3|\beta|}(1+O(|\eta'|^{-1/3})) \qquad \text{for } \alpha>0.$$

Furthermore, setting $l(\eta')=L(\eta')e^{i(4/3)\alpha^{3/2}|\eta'|}$, we obtain

$$l(\eta')=ie^{-2\tau\sqrt{\alpha}}(1+O(\zeta^{-3/2})) \qquad \text{for } \alpha|\eta'|^{2/3}\gg 1.$$

Therefore $(L(1-\chi_\varepsilon)\chi_1)(\eta')\in S^0_{\varepsilon/2,0}$ and $L\chi_\varepsilon$ is a Fourier integral operator with amplitude $(l\chi_\varepsilon)(\eta')\in S^0_{1-\varepsilon,0}$ and with the following singular canonical transformation:

$$\phi_2(y',\eta')=(y_0+2\sqrt{\alpha}\,(1-(1/3)\alpha^2),\ y''-(2/3)\alpha^{3/2}\eta''/|\eta'|,\ \eta'),$$

which is similar to (3.33) in [2].

Now, applying Φ_1^{-1} to $BGv=f$, from (6) through (9) we have

(10) $\qquad \Phi_1^{-1}BG_0v_0+\Phi_1^{-1}B(G_hv_h+G_ev_e)=\Phi_1^{-1}f.$

Here $\Phi_1^{-1}BG_0=\Phi_1^{-1}BG_1q_1+\Phi_1^{-1}BG_2q_2$ and $\Phi_1^{-1}BG_j=\tilde{c}_j(1+L\chi_1)+\tilde{d}_j\mathcal{L}$, where $\tilde{c}_j, \tilde{d}_j\in OPS^0_{1,0}$ and, mod $S^{-1}_{1,0}$, $\tilde{c}_j(y',\eta')=B(x)a_{j0}(x,\eta')$, $\tilde{d}_j(y',\eta')=|\eta'|^{1/3}B(x)b_{j0}(x,\eta')$. According to (H_2) and (H_3), one can take a positive number δ such that, for $\alpha=0$, $R_0(x',\xi')\neq 0$ if $y_0\leqq\bar{y}_0-2\delta$ and $R_0(x',\xi')=0$ if $y_0>\bar{y}_0-\delta$ and $m_1\geqq 2$. We then take q_1, q_2 so that $q_1(y_0)=1$ for $y_0>\bar{y}_0-6\delta$ and $q_2(y_0)=1$ for $y_0<\bar{y}_0-7\delta$. By (4) we may also reduce (10), as in [5], § 5, to the following equation only for v_0:

(11) $\qquad a(1+L\chi_1)q_1v_0+b\mathcal{L}q_1v_0+c(1+L\chi_1)q_2v_0+d\mathcal{L}q_2v_0=f_0,$

where a, b, c and $d\in OPS^0_{1,0}$ are $m_1\times m_1$ matrices. Besides, setting

$$a=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix},\quad b=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix},\quad v_0=\begin{bmatrix}v_1\\ v_2\end{bmatrix}\quad\text{and}\quad f_0=\begin{bmatrix}f_1\\ f_2\end{bmatrix},$$

where a_{11}, b_{11}, v_1 and f_1 are scalar, and denoting by $a_{11}(y',\eta')$ the principal symbol of a_{11} and so on, we have, for $y_0>\bar{y}_0-3\delta$, $a_{12}(y',\eta')=O(\alpha)$, $a_{21}(y',\eta')=O(\alpha)$, $a_{22}(y',\eta')=I_{m_1-1}$ and $b_{11}(y',\eta')=1$. Hereafter I_k stands for the identity matrix of degree k. Moreover by virtue of (H_1) we can assume arg $a_{11}(y',\eta')\subset[-\pi/2, (\pi-\delta_1)/2]$ for $\alpha=0$. Furthermore (H_2) yields, as in [11], p. 540, that, for $y_0<\bar{y}_0-4\delta$, $d(y',\eta')=O(\alpha)$, $a(y',\eta')=I_{m_1}+O(\alpha)$, $c(y',\eta')=I_{m_1}$ and that Re $a(y',\eta')^{-1}$ is positive definite for $\alpha=0$ and $y_0<\bar{y}_0-2\delta$. Finally, when $m_1\geqq 2$, (H_3) implies that $a_{11}(y',\eta')=O(\alpha)$ for $y_0>\bar{y}_0-\delta$.

Now, a basic a priori estimate for (11) is the following:

$$\|\gamma v_0\|_s^2 \leqq C_1\tau^{-1}\|\gamma^{-1}f_0\|_s^2 + O(\|\gamma^{-1}v_0\|_{s-1}^2)$$

for $\tau \gg 1$ and $v_0 \in H^{s+1/3}(R^n)$ with supp $\hat{v}_0(\eta') \subset \{\tau\gamma \ll 1\}$. To prove this we use also

$$\mathrm{Re}\,(\mathcal{L}v, (1+L\chi_1)v) \geqq C_2\tau(\|\gamma\chi_1 v\|^2 + \|\gamma^{-1/2}(1-\chi_1)v\|_{-1/2}^2) - O(\|v\|_{-1/2}^2)$$

for $v \in L^2(R^n)$ with supp $\hat{v} \subset \{\tau^2\alpha \ll 1\}$, where $C_2 > 0$. To deduce the regularity near the hyperbolic region we need the following a priori estimate. Suppose $p(y', \eta') \in S^0_{1,0}$, $0 \leqq p(y', \eta') \leqq 1$ and $p \circ \phi_2(y', \eta') \leqq p(y', \eta')$. Then

$$C_3\tau(\|\gamma p v_1\|_s^2 + \|p v_2\|_s^2) \leqq \|\gamma^{-1} p f_0\|_s^2 + O(\|\gamma v_1\|_{s-\varepsilon_0}^2 + \|v_2\|_{s-\varepsilon_0}^2)$$

for $\tau \gg 1$ and $v_0 \in H^{s+1/3}(R^n)$ such that supp $\hat{v}_0 \subset \{|\eta'|^{-\varepsilon} < \alpha \ll \tau^{-2}\}$ and $WF(v_0) \subset \{y_0 > \bar{y}_0 - \delta\}$, where $\varepsilon_0 = 1/2 - (3/4)\varepsilon$ and C_3 is a positive number independent of p. Furthermore to conclude that $v_0 \in H^\infty(R^n \cap \{y_0 \ll \bar{y}_0\})$, where v_0 is a solution of (11), we use the following: Let $f(y')$ be a distribution in R^n, supported in a compact set $\subset R^n \cap \{y_0 \geqq 0\}$. Then $(1+L)^{-1}(1-\chi_\varepsilon)\chi_1 f \in H^\infty(R^n \cap \{y_0 < -\delta\})$ for any $\delta > 0$. It should be pointed out that $(1+L)^{-1}(1-\chi_\varepsilon)\chi_1$ belongs only to a bad class $OPS^{1/3}_{0,0}$ and hence does not have the pseudolocal property.

References (continued from [I])

[11] Eskin, G.: Comm. in P. D. E., **1**, 521–560 (1976).
[12] Farris, M.: ibid., **6**, 651–687 (1981).
[13] Kubota, K.: (preprint) (1984).
[14] Taylor, M. E.: Singularities in Boundary Value Problems (edited by H. G. Garnir). D. Reidel Publ., pp. 271–316 (1981).