62. *On Semisimple Lie Algebras over Algebraically Closed Fields*

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Let $\mathfrak{g}_C$ be a semisimple Lie algebra over the complex number field C, $\mathfrak{h}_C$ a Cartan subalgebra of it, and Σ the root system of $(\mathfrak{g}_C, \mathfrak{h}_C)$. Let $H^\omega \in \mathfrak{h}_C$ be the element corresponding to a coweight ω, and choose root vectors $X_\alpha (\alpha \in \Sigma)$ in such a way that we get a Chevalley basis from these elements. Denote by $\mathfrak{g}_Z$ and $\mathfrak{h}_Z$ the Lie algebras over the ring of integers Z spanned by $\{X_\alpha, H^\omega\}$ and by $\{H^\omega\}$ respectively. For a ring or a field F, we define $\mathfrak{g}_F = F \otimes_Z \mathfrak{g}_Z$. In this paper, we consider the Lie algebra $\mathfrak{g} = \mathfrak{g}_K$ over an algebraically closed field K, and study its nilpotent classes under the adjoint group G corresponding to $\mathfrak{g}$. Let $p = \mathrm{ch}(K)$ be the characteristic of K. We start from $p=0$, and then study how the situation varies when $p>0$ becomes small.

1. Standard representatives of nilpotent classes. For a subset S of Σ, put $X_S = \sum_{\alpha \in S} X_\alpha \in \mathfrak{g}_Z$. We denote by $\bar{X}$ the element $1 \otimes X \in \mathfrak{g}$ for $X \in \mathfrak{g}_Z$.

Definition. A nilpotent element of the form $\bar{X}_S$ in $\mathfrak{g}$ is called a standard representative (SR) of its class if it satisfies the following conditions. (1) S is linearly independent. (2) S is a Π-system or a $(\Pi, 1)$-system. (3) S is minimal for p $(=\mathrm{ch}(K))$ in the sense that for any $\alpha \in S$, $\bar{X}_{S'}$ with $S' = S \setminus \{\alpha\}$ is no longer conjugate to $\bar{X}_S$.

Here a subset S of Σ is called a Π-system if $\gamma - \gamma' \notin \Sigma$ for any γ, $\gamma' \in S$. It is called a $(\Pi, 1)$-system if it satisfies the following: let $S = \bigcup_j S^j$ be the finest decomposition of S such that $S^i \perp S^j$ for $i \neq j$, then for any j, (i) S^j is a Π-system, or (ii) there exists a pair $\{\alpha, \beta\} \subset S^j$ such that the inner product $(\alpha, \beta) > 0$, and $(\gamma, \gamma') \leqslant 0$ for any other pair $\{\gamma, \gamma'\} \subset S^j$, and that both $S^j \setminus \{\alpha\}$, $S^j \setminus \{\beta\}$ are Π-systems.

We know that in case $p=0$, every nilpotent class has several types of SRs. Moreover different types of such representatives may have their own rights as is recognized from the result at the end of [2].

For $S \subset \Sigma$, its characteristic diagram $\mathrm{ch}(S)$ is defined as follows. To every root $\gamma \in S$, we assign a node, and two nodes γ, $\gamma' \in S$ are connected by k segments or k waved segments if $(\gamma, \gamma') < 0$ or $(\gamma, \gamma') > 0$ respectively, where $k = |\gamma|^2 |\gamma'|^2 / 4 |(\gamma, \gamma')|^2$. Moreover we attach to this diagram the ratios of root lengths for every simple component of Σ. If S is a Π-system, $\mathrm{ch}(S)$ is nothing but the Dynkin diagram of S.

From now on, we assume for simplicity that $\mathfrak{g}$ is simple. Further, for Theorems 1–4, we assume that $\mathfrak{g}$ is not of type F_4 for a technical reason.

Theorem 1. *Assume that $S\subset\Sigma$ satisfies the conditions* (1) *and* (2). *Then S is minimal for $p=0$ if and only if so is it for some or any prime $p>0$ good for* $\mathfrak{g}$.

Theorem 2. *Assume that $S_1, S_2\subset\Sigma$ are both $(\Pi, 1)$-systems satisfying the conditions* (1), (2), *and* (3) *for $p=0$. Then* $\mathrm{ch}(S_1)\cong\mathrm{ch}(S_2)$ *if and only if S_1 is conjugate to S_2 under* $\mathrm{Aut}(\Sigma)$, i.e., $S_1=\sigma S_2$ *for some* $\sigma\in\mathrm{Aut}(\Sigma)$.

In Theorem 2, in case $\mathrm{Aut}(\Sigma)\supsetneqq\mathrm{Int}(\Sigma)=W(\Sigma)$, the Weyl group of Σ, if $S_1\neq wS_2$ for any $w\in W(\Sigma)$ but $S_1=\sigma S_2$ for some $\sigma\in\mathrm{Aut}(\Sigma)\setminus W(\Sigma)$, then the conjugacy classes of $\overline{X}_{S_1}$, $\overline{X}_{S_2}$ are invariant under $\mathrm{Aut}(\mathfrak{g})$ (for good p), so they coincide with each other. Hence we get the following

Theorem 3. *Let p be good for $\mathfrak{g}$. Let $\overline{X}_S$ be a standard representative. If S is a $(\Pi, 1)$-system, then its conjugacy class is determined by the characteristic diagram* $\mathrm{ch}(S)$ *of S.*

In case $p=0$, for a non-trivial nilpotent class $\mathcal{O}$, there corresponds a unique element $H_0\in\mathfrak{h}_Z$ as follows. Take $X\in\mathcal{O}$. There exists a semisimple element H' of $\mathfrak{g}_C$ such that $[H', X]=2X$, $H'\in\mathrm{ad}(X)\mathfrak{g}_C$. Then H_0 is by definition a unique dominant element in $\mathfrak{h}_C$ conjugate to H'. Since $\alpha(H_0)=0$, 1 or 2 for any simple root α, we see that $H_0\in\mathfrak{h}_Z$. Thus, for a class $\mathcal{O}$, we get a gradation $\mathfrak{g}_Z=\sum_{i\in Z}\mathfrak{g}_Z(i)$ by $\mathrm{ad}(H_0)$. This gives a gradation of $\mathfrak{g}$ as $\mathfrak{g}=\sum_{i\in Z}\mathfrak{g}(i)$, $\mathfrak{g}(i)=K\otimes_Z\mathfrak{g}_Z(i)$.

Theorem 4. *Let p be good for $\mathfrak{g}$. Then for every standard representative $\overline{X}_S$ of a class $\mathcal{O}$, there exists a $w\in W(\Sigma)$ such that $\overline{X}_{wS}$ belongs to* $\mathfrak{g}(2)$.

Note that $\overline{X}_{wS}$ is again an SR of $\mathcal{O}$ with $\mathrm{ch}(wS)\cong\mathrm{ch}(S)$.

For every type of $\mathfrak{g}$ except for type F_4, we determined all the SRs $\overline{X}_S$ modulo the conjugacy of S under $W(\Sigma)$, when p is good. This means essentially the determination of possible diagrams $\mathrm{ch}(S)$ of $(\Pi, 1)$-systems, since Π-systems had been studied in [1]. Further the situations can also be studied when p is no longer good.

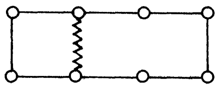
2. Jacobson-Morozov type Theorem. In case $p=0$, any nonzero nilpotent element is conjugate to an SR $X_S\in\mathfrak{g}_Z$. We can take X_S from $\mathfrak{g}_Z(2)$ corresponding to its conjugacy class. Then we can prove by explicit calculation case by case that there exists a vector Y_0 in $\mathfrak{g}_{Z[1/r]}$ such that $[H_0, Y_0]=-2Y_0$, $[X_S, Y_0]=H_0$, where $r=1$, 2 or 3. By tensoring with $1\in K$, and examining coefficients of Y_0 with respect to X_α's, we get the following.

Theorem 5. *Let p be good for $\mathfrak{g}$, and $\neq 2$. Then for any nonzero nilpotent element $X\in\mathfrak{g}$, there exist a semisimple element H, and* *a nilpotent one Y such that*

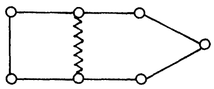
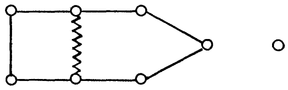
(JM) $\qquad [H, X]=2X, \quad [\mathrm{H}, Y]=-2Y, \quad [X, Y]=H.$

Note that unlike in case p is sufficiently large, the uniqueness assertion for H and Y is no longer valid when p is small enough compared with the Coxeter number of $\mathfrak{g}$ (cf [5]).

For every standard X_S above, we know Y_0 explicitly, and so we know that (JM) is also true for $p=5$ and $\mathfrak{g}$ of type E_8, taking $X=\bar{X}_S$, $H=\bar{H}_0$, $Y=\bar{Y}_0$. (Recall that bad primes for E_8 are 5, 3, 2, and those for E_6 and E_7 are 3, 2.) For $p=3$, and $\mathfrak{g}$ of type E_6, E_7 or E_8, the only case where no Y exists for $X=\bar{X}_S$, $H=\bar{H}_0$, for which (JM) holds, is given by $S=S_1$ for E_8 with $\mathrm{ch}(S_1)$ below. This is also the only case for E_6, E_7 and E_8, where coefficients of Y_0 have a denominator $r=3$.

$\mathrm{ch}(S_1)$: 

Moreover we remark here that the only cases for E_6, E_7 and E_8, where coefficients of Y_0 have a denominator $r=2$, are given by $S=S_2$ for E_7, and $S=S_2$, S_3 for E_8 with $\mathrm{ch}(S_i)$ $(i=2, 3)$ given below. When $p=\mathrm{ch}(K)$ is good for $\mathfrak{g}$, these SRs $\bar{X}_{S_i}$ $(i=1, 2, 3)$ represent conjugacy classes with names $D_8(a_3)$, $D_6(a_1)+A_1$, $D_6(a_1)+2A_1$ respectively.

$\mathrm{ch}(S_2)$:  $\mathrm{ch}(S_3)$: 

3. Adjoint representation on $\mathfrak{g}$. For types E_6, E_7 and E_8, we calculated all elementary divisors of $\mathrm{ad}(X_S)^t$ $(t=1, 2, \cdots)$ for any SR $X_S \in \mathfrak{g}_Z$, taking its conjugate from $\mathfrak{g}_Z(2)$. Then reducing them by mod p, we get the following results.

Theorem 6. *Let $\mathfrak{g}$ be of type E_6, E_7 or E_8, and p be good for $\mathfrak{g}$. Then two nilpotent elements X_1, X_2 are conjugate under G if and only if so are* $\mathrm{ad}(X_1)$, $\mathrm{ad}(X_2)$ *under* $\mathrm{GL}(\mathfrak{g})$.

Note (T. Umeda)**.** Let $\mathfrak{g}$ be classical and $p=0$. Then the assertion of Theorem 6 is true except the cases of type D_{2n} $(n=2, 3, \cdots)$.

Theorem 7. *Let $\mathfrak{g}$ be of classical type or of type E_6, E_7* or E_8. *Let $X_S \in \mathfrak{g}_Z$ be a standard representative for $p=0$, and*

$$\mathrm{ad}(X_S)^N \neq 0, \quad \mathrm{ad}(X_S)^{N+1}=0 \quad \text{on } \mathfrak{g}_Z.$$

Then any prime number $q \geqslant N+1$ does not appear in elementary divisors of $\mathrm{ad}(X_S)^t$, $t \geqslant 1$. *In particular, any good prime does not appear in elementary divisors of* $\mathrm{ad}(X_S)$.

This theorem implies that the Jordan normal form of $\mathrm{ad}(\bar{X}_S)$ on $\mathfrak{g}$ has the same form as for $p=0$ as long as $p=\mathrm{ch}(K) \geqslant N+1$. On the other hand, if $p \leqslant N$, the Jordan normal form varies depending on p. We can determine it completely for every SR $X_S \in \mathfrak{g}_Z$. This gives us many informations about the degeneracy of the mapping $\mathrm{ad}(X_S)$:

$\mathfrak{g}\to\mathfrak{g}$, and conversely about that of the orbit $\mathrm{Ad}(G)\bar{X}_S$. In particular, we see that different types of SRs $\bar{X}_S$ of the same class for p good are no longer conjugate to each other when p becomes bad.

Theorem 8. *Let $\mathfrak{g}$, X_S and N be as in Theorem 7. Then for $X=\bar{X}_S\in\mathfrak{g}$,*

$$\mathrm{ad}(X)^p=0 \quad \text{on } \mathfrak{g} \quad \text{if } [N/2]+1\leqslant p\leqslant N,$$
$$\mathrm{ad}(X)^p\neq 0 \quad \text{on } \mathfrak{g} \quad \text{if } p\leqslant [N/2].$$

We denote by $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{l})$ the adjoint representation of $\mathfrak{g}$ restricted to a subalgebra $\mathfrak{l}$ of $\mathfrak{g}$.

Theorem 9. *Let $\mathfrak{g}$, X_S and N be as in Theorem 7. Assume that there exist an H semisimple, a Y nilpotent in $\mathfrak{g}$ for which* (JM) *holds together with $X=\bar{X}_S$. Let $\mathfrak{l}$ be the subalgebra of $\mathfrak{g}$, isomorphic to $\mathfrak{sl}_2(K)$, generated by X, H and Y. Then $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{l})$ is not completely reducible for $2<p\leqslant[N/2]$. If H, Y come from H_0, Y_0 as $H=\bar{H}_0$, $Y=1\otimes Y_0$, then $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{l})$ is completely reducible for $p\geqslant N+1$, and not for $2<p\leqslant N$.*

This theorem extends partially Jacobson's result in [3].

References

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