59. The Unit Indices of Imaginary Abelian Number Fields

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(Communicated by Shokichi IYANAGA, M. J. A., June 12, 1984)

1. **Introduction.** Let K be an imaginary abelian number field with conductor f and K_0 its maximal real subfield. Let E_K, E_{K_0} be the groups of units of K, K_0 respectively. Let W_K be the group of roots of unity in K. Then the unit index (the Hasse index) Q_K of K is defined as

$$Q_K=[E_K : W_K E_{K_0}].$$

As Hasse [1] showed, $Q_K=1$ or 2. He investigated the properties of the unit index, which, however, do not suffice to determine it in many cases.

Hasse [1] proved in Satz 23 that if f is a power of a prime, then $Q_K=1$. The aim of this note is to determine the unit index Q_K of K of certain types whose conductor f is a product of two or three prime powers. As a consequence we construct some counterexamples to Satz 29 of [1].

In the following, we let p, q be distinct odd primes and a, b and n positive integers.

Notations. $\boldsymbol{Q}$ denotes the field of rational numbers. ζ_f denotes a primitive f-th root of unity. χ_4, χ_p denote odd Dirichlet characters with conductor 4, p respectively. ψ_{2^n} denotes even Dirichlet character with conductor 2^n. For any abelian number field L, we denote by $X(L)$ the character group corresponding to L, and $h(L)$ the ideal class number. $\langle x, y, \cdots \rangle$ denotes the group generated by $x, y, \cdots$. $a|b$ (resp. $a^n \| b$) means that a divides b (resp. a^n divides b and a^{n+1} does not divide b).

2. **Results.** In the following we will use Sätze of [1] except Satz 29.

First we treat the case $f=4p^a$ and $f=p^a q^b$.

Theorem 1. *Let $f=4p^a$ or $f=p^a q^b$. Then $Q_K=2$ if and only if the relative degree $[\boldsymbol{Q}(\zeta_f) : K]$ is odd.*

For the proof we need Sätze of [1] and

Lemma 1. *If k is an imaginary subfield of K with odd relative degree $[K : k]$, then $Q_k=Q_K$ where Q_k is the unit index of k.*

Next we consider the case $f=2^n p^a$ $(n\geqq 3)$. If $2\|(p-1)$, we can

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determine all the unit indices Q_K for such conductor f by Sätze of [1]. If $2^2\|(p-1)$, then so we can do by Sätze of [1] and the following

Theorem 2. *Suppose* $2^2\|(p-1)$. *If* K *is the imaginary abelian number field corresponding to* $X(K)=\langle\chi_4, \psi_{2^n}\chi_p^{(p-1)/2}\rangle$ *or* $\langle\chi_4\psi_{2^n}, \chi_4\chi_p^{(p-1)/2}\rangle$, *then* $Q_K=1$. *More precisely, there exists a system of fundamental units of* K_0 *with arbitrary signatures.*

Finally we assume that $2^3|(p-1)$. We treat here only the case $n=3$, i.e., $f=8p$ because the case $n\geqq 4$ is complicated. First we notice the following

Proposition 1. *Suppose* $2^3|(p-1)$. *Let* ε *be the fundamental unit of* $\boldsymbol{Q}(\sqrt{2p})$. *If* $2\|h(\boldsymbol{Q}(\sqrt{2p}))$, *then* $N\varepsilon=+1$.

The converse of Proposition 1 is not always true. Under the condition that $2\|h(\boldsymbol{Q}(\sqrt{2p}))$, we can determine all the unit indices Q_K of K with conductor $f=8p$, $2^3|(p-1)$, by means of Sätze of [1] and the following Theorem 3 and 4.

Theorem 3. *Suppose* $2^e\|(p-1)$, $e\geqq 3$. *For each* s, $s=2, 3, \cdots, e-1$, *let* k_s *be the imaginary abelian number fields corresponding to* $X(k_s)=\langle\chi_4, \psi_8\chi_p^{(p-1)/2^s}\rangle$ *or* $\langle\chi_4\psi_8, \chi_4\chi_p^{(p-1)/2^s}\rangle$. *If* $2\|h(\boldsymbol{Q}(\sqrt{2p}))$, *then* $Q_{k_s}=2$ *for each* s.

Theorem 4. *Suppose* $2^3|(p-1)$. *Let* $k=\boldsymbol{Q}(\sqrt{-1}, \sqrt{2p})$ *or* $k=\boldsymbol{Q}(\sqrt{-2}, \sqrt{-p})$. *Then* $Q_k=2$ *if and only if* $N\varepsilon=+1$ *where* ε *is the fundamental unit of* $\boldsymbol{Q}(\sqrt{2p})$.

The proof of Theorem 3 depends on the following

Lemma 2. *Let* l *be a prime number. Let* k *be a real number field of finite degree over* $\boldsymbol{Q}$ *and* K *a real cyclic extension of* k *of degree* l^m, $m\geqq 2$. *Let* F *be the intermediate field of* K/k *such that* $[F:k]=l$. *Suppose that there exist two distinct prime ideals of* k *which are totally ramified in* K/k *while any other prime ideal is unramified in* K/k. *If* $l\nmid h(k)h(F)$, *then* $l\nmid h(K)$.

In the case $f=4p^aq^b$, we have analogous results to the case $f=8p^a$. For example, we obtain the following proposition similar to Theorem 4.

Proposition 2. *Let* p, q *be distinct odd primes such that* $p\equiv q\equiv 1 \pmod 4$. *Let* $k=\boldsymbol{Q}(\sqrt{-p}, \sqrt{-q})$. *Then* $Q_k=2$ *if and only if* $N\varepsilon=+1$ *where* ε *is the fundamental unit of* $\boldsymbol{Q}(\sqrt{pq})$.

3. Remarks. (1) Hasse [1] tabulated the unit indices Q_K and the relative class number h_K^* of imaginary abelian number fields K with conductor $f\leqq 100$. We prolonged his table for $100<f\leqq 200$ ([2]).

(2) Hasse asserted in Satz 29 of [1]: if k is an imaginary subfield of K, then Q_k divides Q_K. However this divisibility does not hold true in general. In fact, we obtain some counterexamples as follows.

Example 1. Let p and k_s be as in Theorem 3. Let K_s be the imaginary abelian number fields corresponding to $X(K_s)=\langle\chi_4, \psi_8, \chi_p^{(p-1)/2^s}\rangle$. Then $K_s \supseteqq k_s$ and $Q_{K_s}=1$, and $Q_{k_s}=2$ for each $s=2, 3, \cdots, e-1$, if $2 \| h(\boldsymbol{Q}(\sqrt{2p}))$.

Example 2. Let p be an odd prime such that $2^3 | (p-1)$. Let $K=\boldsymbol{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{p})$ and $k=\boldsymbol{Q}(\sqrt{-1}, \sqrt{2p})$ or $k=\boldsymbol{Q}(\sqrt{-2}, \sqrt{-p})$. Then $Q_K=1$, and $Q_k=2$ if $N\varepsilon=+1$ where ε is the fundamental unit of $\boldsymbol{Q}(\sqrt{2p})$.

For conductor $f=4pq$, we have analogous counterexamples to Satz 29.

References

[1] H. Hasse: Über die Klassenzahl abelscher Zahlkörper. Akademie Verlag, Berlin (1952).

[2] K. Yoshino and M. Hirabayashi: On the Relative Class Number of the Imaginary Abelian Number Field I, II. Memoirs of the College of Liberal Arts, Kanazawa Medical University, vols. 9–10 (1981, 1982).