# 49. On Obstructions of Infinitesimal Lifting

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Let $X$ be the analytic submanifold of an algebraic manifold $P$ defined by an ideal sheaf $I_X$. Let $E$ be a holomorphic vector bundle on $P$ and $E|_X$ its restriction to $X$. Then, we consider the formal completion $\hat{P}$ of $P$ and the formal completion $\hat{E}$ of $E$ along $X$.

Assume that a holomorphic section $\bar{\sigma}$ of $E|_X$ is given. Then, it is not easy to find a criterion in terms of $\bar{\sigma}$ and $E|_X$ for liftability of the section $\bar{\sigma}$ to a section of $\hat{E}$. Getting a lifting $\hat{\sigma}$ in $\hat{E}$ of $\bar{\sigma}$ is equivalent to constructing a system $\{\sigma_{(\mu)}\}_{\mu=1}^{\infty}$ of sections of $\{O_P(E)\otimes O_P/I_X^{\mu}\}$ such that (1) $\sigma_{(1)}=\bar{\sigma}$, (2) $\pi_{\mu+1}(\sigma_{(\mu+1)})=\sigma_{(\mu)}$ for every $\mu\geqq 1$ where $\pi_{\mu+1}$; $O_P(E)\otimes O_P/I_X^{\mu+1}\to O_P(E)\otimes O_P/I_X^{\mu}$ is a natural projection.

When $\sigma_{(\mu)}$ is given, the obstruction for finding a lifting $\sigma_{(\mu+1)}$ of $\sigma_{(\mu)}$ is $\delta_\mu(\sigma_{(\mu)})$ which appears in the following sequence,

$$0\longrightarrow H^0(O_X(S^{(\mu)}(N^*_{X/P})\otimes E|_X))\longrightarrow H^0(O_P(E)\otimes O_P/I_X^{\mu+1})\longrightarrow H^0(O_P(E)\otimes O_P/I_X^{\mu})\xrightarrow{\delta_\mu} H^1(O_X(S^{(\mu)}(N^*_{X/P})\otimes E|_X))\longrightarrow\cdots$$

where $N^*_{X/P}$ is the conormal bundle of $X$ in $P$. $\delta_\mu(\sigma_{(\mu)})$ is called "$\mu$-th obstruction". The following fact should be pointed out here: Even if $\delta_{(\mu_0)}(\sigma_{(\mu_0)})$ does not vanish for some system $\{\sigma_{(\mu)}\}_{\mu=1}^{\mu_0}$ in a course of constructing a system, this does not mean the impossibility of lifting of $\bar{\sigma}$. We may find another system $\{\tau_{(\mu)}\}_{\mu=1}^{\mu_0}$ with $\delta_{(\mu_0)}(\tau_{(\mu_0)})=0$ which extends $\bar{\sigma}$.

Now, let us consider a following special case which is useful in the study of the defining equations. Let $P$ be a projective space $\boldsymbol{P}^N(\boldsymbol{C})$ and $E$ be $\Omega_P^1(\nu)=\Omega_P^1\otimes O_P(1)^{\otimes\nu}$.

**Theorem.** *Assume the embedding of $X$ into $P=\boldsymbol{P}^N$ is projectively normal. Then the given section $\bar{\sigma}$ of $H^0(X, \Omega_P^1(\nu)\otimes O_X)$ can be lifted to a section of $H^0(\hat{P}, \hat{\Omega}_P^1(\nu))$ if and only if the first obstruction $\delta_1(\bar{\sigma})$ vanishes.*

The full proof will be given in [1]. Here, we shall give only a rough sketch of the proof. First, we make a special vector subspace $V_0$ of $H^0(X, \Omega_P^1(\nu)\otimes O_X)$ and another operator

$$\mathfrak{D}\,;\ H^0(X, \Omega_P^1(\nu)\otimes O_X)\longrightarrow H^1(X, O_X(N^*_{X/P})\otimes\Omega_P^1(\nu))$$

by using special property of $\Omega_P^1(\nu)$. And then, we show that on $V_0$, $\mathfrak{D}$

equals $\delta_1$ and any section of $H^0(X, \Omega^1_P(\nu)\otimes O_X)$ can be lifted modulo $V_0$. Then using a good nature of $\mathfrak{D}$ and projective normality, the above statement is affirmative for $V_0$. The difficulty of the problem comes from the fact that $\delta_1$ is not equal to $\mathfrak{D}$ on the whole space $H^0(X, \Omega^1_P(\nu)\otimes O_X)$.

**Remark 1.** Every algebraic manifold has a projectively normal embedding. Every manifold $X$ embedded in $\boldsymbol{P}^N$ has, however, an obstructed section in $H^0(X, \Omega^1_P(\nu)\otimes O_X)$ for some $\nu$.

**Remark 2.** When we do not assume that the projective normality of $X$, the theorem is not true.

As a corollary of the proof, we can get the following fact.

**Corollary.** *Assume that $X$ is embedded in $P=\boldsymbol{P}^N$ projectively normally. Then the normal bundle $N_{X/P}$ splits into direct sum of extendable line bundles $O_X(d_1)\oplus\cdots\oplus O_X(d_r)$ if and only if $X$ is complete intersection of type $(d_1, \cdots, d_r)$.*

For another application, see [1].

## Reference

[1] T. Usa: On obstructions of infinitesimal lifting (in preparation).