44. On a Multi-valued Differential Equation: An Existence Theorem

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1. Introduction. Let $\mathfrak{H}$ be a real Hilbert space of finite dimension and a correspondence (=multi-valued mapping) $\Gamma:[0, T]\times\mathfrak{H}\twoheadrightarrow\mathfrak{H}$ is assumed to be given. The compact interval $[0, T]$ is endowed with the usual Lebesgue measure dt. In this paper, we shall establish a sufficient condition which assures the existence of solutions of a multi-valued differential equation of the form:

$$\begin{cases}\dot{x}(t)\in\Gamma(t, x(t))\\ x(0)=a,\end{cases}\tag{$*$}$$

where a is a fixed vector in $\mathfrak{H}$. The author is very much indebted to the works of Attouch-Damlamian [2] and Castaing [3], which treat the closely related problems. The purpose of this paper is to reconstruct their theories in the framework of the Sobolev space $\mathcal{W}^{1,2}$. We shall then proceed, in another paper [7], to examining some variational problems governed by a multi-valued differential equation like ($*$). A couple of existence theorems of optimal solutions for them will be shown there.

2. Assumption. Let us begin by specifying some assumptions imposed on the correspondence Γ.

Assumption 1. Γ is compact-convex-valued; i.e. $\Gamma(t, x)$ is a non-empty, compact and convex subset of $\mathfrak{H}$ for all $t\in[0, T]$ and all $x\in\mathfrak{H}$.

Assumption 2. The correspondence $x\twoheadrightarrow\Gamma(t, x)$ is upper hemi-continuous (abbreviated as u.h.c.) for each fixed $t\in[0, T]$.

Assumption 3. The correspondence $t\twoheadrightarrow\Gamma(t, x)$ is measurable for each fixed $x\in\mathfrak{H}$. For the concept of "measurability" of a correspondence, see [6] Chap. 6.

Assumption 4. There exists $\psi\in L^2([0, T], \boldsymbol{R}_+)$ such that $\Gamma(t, x)\subset S_{\psi(t)}$ for every $(t, x)\in[0, T]\times\mathfrak{H}$, where $S_{\psi(t)}$ is the closed ball in $\mathfrak{H}$ with the center 0 and the radius $\psi(t)$.

3. Preliminary Lemmas.

Lemma 1 (Castaing [3]). *Let Γ be a correspondence which satisfies Assumption 1-3. And let $x:[0, T]\to\mathfrak{H}$ be any measurable mapping. Then there exists a closed-valued measurable correspondence $\Sigma:[0, T]\twoheadrightarrow\mathfrak{H}$ such that $\Sigma(t)\subset\Gamma(t, x(t))$ for all $t\in[0, T]$.*

Lemma 2. *Let A be a non-empty, convex and compact subset of*

$\mathfrak{H}$ and define a subset $\mathcal{X}$ of the Sobolev space $\mathcal{W}^{1,2}([0, T], \mathfrak{H})$ by

$$\mathcal{X}=\{x \in \mathcal{W}^{1,2} \mid \|\dot{x}(t)\| \leqq \psi(t) \text{ a.e. and } x(0) \in A\}.$$

Then $\mathcal{X}$ is a non-empty, convex and weakly compact subset of $\mathcal{W}^{1,2}$.

Lemma 3. *Suppose that the Assumptions 1–2 are satisfied and let (t^*, x^*) be any element of $[0, T]\times\mathfrak{H}$. If we define a subset $K(t^*; x^*, \varepsilon)$ of $[0, T]\times\mathfrak{H}$ by*

$$K(t^*; x^*, \varepsilon)=\{(t, x) \in [0, T]\times\mathfrak{H} \mid \|x-x^*\| \leqq \varepsilon,\ t=t^*\}$$

for each $\varepsilon>0$, then the following relation holds good:

$$\Gamma(t^*, x^*)=\bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, \Gamma(K(t^*; x^*, \varepsilon)).$$

Since the proofs of Lemmas 2, 3 are quite easy, we may omit them.

Lemma 4. *Suppose that the Assumptions 1, 2 and 4 are satisfied. Let A be a non-empty, convex and compact subset of $\mathfrak{H}$. Then the set*

$$H=\{(a, x, y) \in A\times\mathcal{X}\times\mathcal{X} \mid \dot{y}(t) \in \Gamma(t, x(t)) \text{ a.e. and } x(0)=y(0)=a\}.$$

is weakly compact in $A\times\mathcal{X}\times\mathcal{X}$.

Proof. Since $A\times\mathcal{X}\times\mathcal{X}$ is weakly compact, it is enough to show that H is weakly closed. Since $\mathcal{W}^{1,2}$ is a separable Hilbert space (cf. [1] Theorem 3.5, p. 47), the bounded set $\mathcal{X}$ endowed with the weak topology is metrizable (cf. [6] p. 357). Thus we can safely use a sequence-argument to check the closedness of H.

Let $q_n\equiv\{(a_n, x_n, y_n)\}$ be a sequence in H which weakly converges to some $q^*\equiv(a^*, x^*, y^*) \in A\times\mathcal{X}\times\mathcal{X}$. In order to check $q^* \in H$, we have only to show that $\dot{y}^*(t) \in \Gamma(t, x^*(t))$ a.e. Since $\{\dot{y}_n\}$ weakly converges to $\dot{y}^*$ in L^2, there exist, for each $j \in N$, some finite elements

$$\dot{y}_{n_j+1},\ \dot{y}_{n_j+2},\ \cdots,\ \dot{y}_{n_j+m(j)}$$

in $\{\dot{y}_n\}$ and

$$\alpha_{ij}\geqq 0, \quad 1\leqq i\leqq m(j), \quad \sum_{i=1}^{m(j)} \alpha_{ij}=1$$

such that

$$\begin{cases} \|\dot{y}^*-\sum_{i=1}^{m(j)} \alpha_{ij}\dot{y}_{n_j+i}\|_{L^2}\leqq 1/j \\ n_{j+1}\geqq n_j+m(j). \end{cases}$$

If we denote

$$\eta_j(t)=\sum_{i=1}^{m(j)} \alpha_{ij}\dot{y}_{n_j+i}(t),$$

then

$$\eta_j(t) \in \mathrm{co}\,(\textstyle\bigcup_{i=1}^{m(j)} \Gamma(t, x_{n_j+i}(t))).$$

And there exists a subsequence (no change in notation) of $\{\eta_j\}$ such that

$$(1) \qquad \eta_j(t)\to\dot{y}^*(t) \quad \text{a.e.}$$

On the other hand, since $\{x_n\}$ weakly converges to x^* in $\mathcal{W}^{1,2}$, there exists a subsequence (no change in notation) of $\{x_n\}$ such that

$$(2) \qquad x_n(t)\to x^*(t) \quad \text{a.e.}$$

Hence, for any fixed $t \in [0, T]$, there exists some $n_0(\varepsilon) \in N$ such that $\|x_n(t)-x^*(t)\|\leqq\varepsilon$ for all $n\geqq n_0(\varepsilon)$, i.e. $(t, x_n(t)) \in K(t; x^*(t), \varepsilon)$ for all $n\geqq n_0(\varepsilon)$. It follows that $\eta_j(t) \in \mathrm{co}\,\Gamma(K(t; x^*(t), \varepsilon))$ for sufficiently large

j. Passing to the limit, we have

(3) $$\dot{y}^*(t) \in \overline{\mathrm{co}}\, \Gamma(K(t\,; x^*(t), \varepsilon)) \quad \text{a.e.}$$

by (1). Since the relation (3) holds good for every $\varepsilon>0$, we can conclude that

(4) $$\dot{y}^*(t) \in \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, \Gamma(K(t\,; x^*(t), \varepsilon)) = \Gamma(t, x^*(t)).$$

The equality in (4) is guaranteed by Lemma 3. Hence $(a^*, x^*, y^*) \in H$. Q.E.D.

4. Main Theorem. We are now going to find out a solution of $(*)$ in the Sobolev space $\mathcal{W}^{1,2}$. Define a set $\Delta(a)$ in $\mathcal{W}^{1,2}$ by $\Delta(a) = \{x \in \mathcal{W}^{1,2} \mid x \text{ satisfies } (*) \text{ a.e.}\}$ for a fixed $a \in \mathfrak{H}$. The following theorem tells us that $\Delta(a) \neq \phi$ and that Δ depends continuously, in some sense, upon the initial value a.

Theorem. *Suppose that Γ satisfies the Assumptions* 1–4, *and let A be a non-empty, convex and compact subset of $\mathfrak{H}$. Then*

(i) *$\Delta(a^*) \neq \phi$ for any $a^* \in A$, and*

(ii) *the correspondence $\Delta: A \twoheadrightarrow \mathcal{W}^{1,2}$ is compact-valued and u.h.c. on A, in the weak topology for $\mathcal{W}^{1,2}$.*

Proof. (i) Fix any $a^* \in A$. If we define a set $\mathcal{X}' \subset \mathcal{X}$ by $\mathcal{X}' = \{x \in \mathcal{X} \mid x(0) = a^*\}$, then $\mathcal{X}'$ is convex and weakly compact in $\mathcal{W}^{1,2}$. Furthermore we define a correspondence $\Phi: \mathcal{X}' \twoheadrightarrow \mathcal{X}'$ by $\Phi(x) = \{y \in \mathcal{X} \mid \dot{y}(t) \in \Gamma(t, x(t)) \text{ a.e.}\}$. Then $\Phi(x) \neq \phi$ for every $x \in \mathcal{X}'$, which can be proved as follows. Let x be any element of $\mathcal{X}'$. Then, by Lemma 1, there exists some closed-valued measurable correspondence $\Sigma: [0, T] \twoheadrightarrow \mathfrak{H}$ such that $\Sigma(t) \subset \Gamma(t, x(t))$ for all $t \in [0, T]$. By the Measurable Selection Theorem (cf. [6] Chap. 6, § 5), there exists a measurable mapping $\sigma: [0, T] \to \mathfrak{H}$ such that

$$\sigma(t) \in \Sigma(t) \subset \Gamma(t, x(t)) \qquad \text{for all } t \in [0, T].$$

If we define $y: [0, T] \to \mathfrak{H}$ by

$$y(t) = a^* + \int_0^t \sigma(s)ds,$$

then it is clear that $y \in \Phi(x)$. Hence $\Phi(x) \neq \phi$. It can easily be checked that Φ is convex-compact-valued. If we define the a^*-section H_{a^*} of H (in Lemma 4) by $H_{a^*} = \{(a, x, y) \in H \mid a = a^*\}$, then H_{a^*} is obviously weakly compact in $A \times \mathcal{X} \times \mathcal{X}$. And the graph $G(\Phi)$ of Φ is expressed as $G(\Phi) = \mathrm{proj}_{\mathcal{X}\times\mathcal{X}} H_{a^*}$, the projection of $G(\Phi)$ into $\mathcal{X} \times \mathcal{X}$, which is also closed.

Summing up——Φ is a convex-compact-valued and u.h.c. Applying now the Ky-Fan's Fixed-Point Theorem ([5]) to the correspondence Φ, we obtain an $x^* \in \mathcal{X}'$ such that $x^* \in \Phi(x^*)$; i.e.

$$\dot{x}^*(t) \in \Gamma(t, x^*(t)) \quad \text{and} \quad x^*(0) = a^*.$$

This proves (i).

(ii) Since the compactness of $\Delta(a)$ can be verified by applying again the Mazur's theorem and making use of the Assumptions 1–2, we may omit the details. Hence we have only to show the u.h.c. of Δ. However it is also obvious because the graph $G(\Delta)$ of Δ can be expressed as

$$G(\Delta)=\mathrm{proj}_{A\times\mathcal{X}}\{(a, x, y)\in H \mid x=y\},$$

which is closed in $A\times\mathcal{X}$. Q.E.D.

Remark. Our Theorem does not necessarily hold in the case $\mathfrak{H}$ is a Hilbert space of infinite dimension even if it is separable. However if we restrict the domain of Γ to the product of $[0, T]$ and *some compact subset of* $\mathfrak{H}$, then the theorem revives again for the separable Hilbert space $\mathfrak{H}$ of infinite dimension.

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