# 43. A Markov Process Associated with a Porous Medium Equation

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**1. Introduction.** In this note we will firstly show the existence of a Markov process $X$ such that the density $u$ of the distribution $P(X(t)\in dx)$ is a weak solution of the Cauchy problem

(1.1a) $$u_t=(1/2)(u^\alpha)_{xx},\qquad t>0,\ x\in \boldsymbol{R},$$

(1.1b) $$u(0,x)=u_0(x),\qquad x\in \boldsymbol{R},$$

for each $\alpha>1$. The equation (1.1a) is called a *porous medium equation*. The existence, uniqueness and properties of the weak solution were studied analytically by Kalashnikov-Oleinik-Yui-lin, Aronson, Kamin and others.

Next we will show that the Markov process $X$ satisfies a stochastic differential equation with a constraint condition about the distribution of $X(t)$. Up to here we do not appeal to any analytic results. Finally we give an example of a process starting from a single point, which is not covered by our above results. To do this we use an explicit solution of (1.1a) given by Barenblatt and Pattle together with an analytic result about the uniqueness of the weak solution of (1.1).

**2. Markov process.** We consider the following conditions for the initial function $u_0$ of (1.1b):

$$(A)\begin{cases} u_0 & \text{is a probability density,}\\ u_0^\alpha & \text{is Lipschitz continuous, and}\\ \int_{\boldsymbol{R}} x^4 u_0(x)dx & \text{is finite.}\end{cases}$$

We will construct a Markov process associated with (1.1) using the following Markov chains. Let $\Omega=\boldsymbol{Z}^N$, and $S_n(\omega)=\omega_n$ for $\omega=(\omega_0,\omega_1,\cdots)\in\Omega$. For each $h>0$, let $P_h$ be the Markov measure on $\Omega$ characterized by the properties that

(2.1a) $$P_h(S_{n+1}=j\,|\,S_n=j_n,\cdots,S_0=j_0)=P_h(S_{n+1}=j\,|\,S_n=j_n),$$

(2.1b) $$P_h(S_{n+1}=j\,|\,S_n=j)=1-\{P_h(S_n=j)\}^{\alpha-1},$$
$$P_h(S_{n+1}=j+1\,|\,S_n=j)=\{P_h(S_n=j)\}^{\alpha-1}/2,$$
$$P_h(S_{n+1}=j-1\,|\,S_n=j)=\{P_h(S_n=j)\}^{\alpha-1}/2,$$

(2.1c) $$P_h(S_0=j)=c(h)^{-1}u_0(jh)h,$$

where $c(h)=\sum_{j\in \boldsymbol{Z}}u_0(jh)h$. Let $C$ be the set of all continuous functions $w:[0,\infty)\to\boldsymbol{R}$, and $\mathcal{F}$ the $\sigma$-field generated by all cylinder sets in $C$.

Let $X_h$ be the $C$-valued random variable on $(\Omega, P_h)$ such that, for each $\omega \in \Omega$, $X_h(\omega)$ is the polygonal function whose value at a point $t \geqq 0$ is

(2.2) $$X_h(t, \omega) = hS_{[t/\tau]}(\omega) + h(t/\tau - [t/\tau])\{S_{[t/\tau]+1}(\omega) - S_{[t/\tau]}(\omega)\},$$

where $\tau = h^{\alpha+1}$. Let $P^h$ be the probability measure on $(C, \mathcal{F})$ such that $P^h(A) = P_h(X_h \in A)$ for all $A \in \mathcal{F}$.

**Theorem 1.** *Under the assumption (A), the family of the probability measures $\{P^h ; h>0\}$ is tight.*

It follows that there exist a subsequence $\{h'\}$ of $\{h\}$ which converges to zero and a probability measure $P$ on $C$ such that $P^{h'}$ converges to $P$ weakly. Set $X(t, w) = w(t)$ for $(t, w) \in [0, \infty) \times C$.

**Theorem 2.** *Under the assumption (A), the process $X = \{X(t)\}$ on $(C, P)$ is a Markov process. Further the distribution $P(X(t) \in dx)$ has a density $u(t, x)$ which is a weak solution of* (1.1).

**Definition.** A function $u$ is called a *weak solution* of (1.1) if $u$ satisfies

(2.3a) $\quad u \in L^1([0, T] \times R) \cap L^\infty([0, T] \times R)$ for all $T > 0$, and

(2.3b) $$\int_0^\infty dt \int_R \left(\psi_t u + \frac{1}{2}\psi_{xx} u^\alpha\right) dx + \int_R \psi(0, x) u_0(x) dx = 0$$

for all $\psi \in C_0^\infty([0, \infty) \times R)$.

We note that the assertion of the existence of a weak solution is a part of Theorem 2. Clearly the initial function $u_0$ becomes the density of the distribution $P(X(0) \in dx)$.

**3. Stochastic differential equation.** We give an expression of the Markov process $X = \{X(t)\}$ on $(C, P)$ constructed in § 2 by a stochastic differential equation. Let $\{\mathcal{F}_t\}$ be the $\sigma$-field generated by $\{X(s) ; s \leqq t\}$ and all $P$-null sets.

**Lemma.** *Under the assumption (A), both processes $\{X(t)\}$ and*

$$\left\{X(t)^2 - \int_0^t u(s, X(s))^{\alpha-1} ds\right\}$$

*are $\{\mathcal{F}_t\}$-martingales.*

Hence we see that

$$B(t) = \int_0^t u(s, X(s))^{-(\alpha-1)/2} dX(s)$$

is an $\{\mathcal{F}_t\}$-Brownian motion. Therefore we have

**Theorem 3.** *Under the assumption (A), the Markov process $X = \{X(t)\}$ on $(C, P)$ satisfies the following stochastic differential equation*

(3.1) $$\begin{cases} X(t) = X(0) + \int_0^t u(s, X(s))^{(\alpha-1)/2} dB(s), \\ P(X(t) \in dx) = u(t, x) dx. \end{cases}$$

We may call the solution of (3.1) *a Markov process associated with a porous medium equation* (1.1).

**4. Another case.** In this section we are concerned with a

Markov process starting from a single point, which is not contained in our previous argument. Barenblatt [3] and Pattle [7] discovered an explicit solution of the equation (1.1a). The one starting from zero is of the form

$$
(4.1)\qquad u_{(\alpha)}(t,x)=\begin{cases}(Lt)^{-\beta}\{1-(Jt)^{-2\beta}x^2\}^{1/(\alpha-1)}, & |x|<(Jt)^{\beta},\\ 0, & |x|\geqq(Jt)^{\beta},\end{cases}
$$

where

$$
\beta=(\alpha+1)^{-1},\quad L=A^2\alpha(\alpha+1)/(\alpha-1),\quad J=A^{1-\alpha}\alpha(\alpha+1)/(\alpha-1),
$$

$$
A=\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\alpha}{\alpha-1}\right)\Big/\Gamma\left(\frac{1}{2}+\frac{\alpha}{\alpha-1}\right).
$$

As $\alpha\to 1$, the function (4.1) becomes the elementary solution of the heat equation:

$$
\lim_{\alpha\downarrow 1}u_{(\alpha)}(t,x)=(2\pi t)^{-1/2}\exp(-x^2/2t).
$$

Using the function (4.1), we can construct a solution of the stochastic differential equation (3.1) starting from zero as follows. For each $\varepsilon>0$, let $P^{(\varepsilon)}$ be the Markov measure on $(\mathcal{C},\mathcal{F})$ stated in Theorem 2 with the initial function $u_0(x)=u_{(\alpha)}(\varepsilon,x)$. By the uniqueness of the weak solution of the equation (1.1) ([8]), we see that $P^{(\varepsilon)}(X(t)\in dx)=u_{(\alpha)}(t+\varepsilon,x)dx$ where $X(t,w)=w(t)$ for $w\in\mathcal{C}$. From the tightness of the family $\{P^{(\varepsilon)}:\varepsilon>0\}$, there exist a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ and a probability measure $P$ on $(\mathcal{C},\mathcal{F})$ such that $P^{(\varepsilon')}$ converges to $P$ weakly. Let $\mathcal{F}_t$ be the $\sigma$-field generated by $\{X(s):s\leqq t\}$ and all $P$-null sets. Then $X=\{X(t)\}$ is an $\{\mathcal{F}_t\}$-martingale and

$$
B(t)=\int_0^t u_{(\alpha)}(s,X(s))^{-(\alpha-1)/2}dX(s)
$$

is an $\{\mathcal{F}_t\}$-Brownian motion. It is shown that

$$
(4.2)\qquad \begin{cases}X(t)=\displaystyle\int_0^t u_{(\alpha)}(s,X(s))^{(\alpha-1)/2}dB(s),\\ P(X(t)\in dx)=u_{(\alpha)}(t,x)dx.\end{cases}
$$

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