23. The Exponential Calculus of Microdifferential Operators of Infinite Order. III

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1. Introduction. In this note we calculate r and c satisfying

$$:ae^p::be^q:=:ce^r:. \tag{1.1}$$

Here a, b, p, q are given formal symbols (see [1]–[4] for the notation). When $a=b=1$, p and q are symbols, such r and c are computed in [2] (cf. [3], [4]). In our present formula, we can take a, b, p, and q to be formal symbols, that is, infinite sums of symbols which satisfy some conditions.

2. Double formal symbols. Let X be an open set in $C^n=\{x=(x_1,\cdots,x_n);\ x_j\in C,\ 1\le j\le n\}$, $\mathring{x}^*$ a point in the cotangent bundle $T^*X\simeq X\times C^n=\{(x,\xi)\in X\times C^n\}$ of X.

Definition 1. Let Ω be a conic neighborhood of $\mathring{x}^*$ in T^*X. Let

$$P(t_1,t_2;x,\xi)=\sum_{j,k=0}^{\infty} t_1^j t_2^k P_{jk}(x,\xi) \tag{2.1}$$

be a formal power series in (t_1, t_2) with coefficients $P_{jk}(x,\xi)$ ($j,k=0,1,2,\cdots$) holomorphic in Ω. The formal series $P(t_1,t_2;x,\xi)$ is said to be a double formal symbol defined in Ω if for any $\Omega'\subset\subset\Omega$ there exist constants d, A which satisfy the following conditions:

(a) $0<d$, $0<A<1$.

(b) For each $h>0$ there is a constant $C>0$ such that

$$|P_{jk}(x,\xi)|\le CA^{j+k}\exp(h|\xi|) \tag{2.2}$$

for all $j,k=0,1,2,\cdots$; $(x,\xi)\in\Omega'$ satisfying $|\xi|\ge(j+k+1)d$.

The space of all double formal symbols defined in Ω is denoted by $\hat{S}_2(\Omega)$, which is a commutative ring under the addition and the product to be those of formal power series. Set $\hat{S}(\Omega)=\hat{S}_1(\Omega)=\hat{S}_2(\Omega)|_{t_2=0}$, then $\hat{S}(\Omega)$ is the ring of all formal symbols defined in Ω ([2], Def. 1; here we consider $t=t_1$).

Definition 2. A double formal symbol

$$P(t_1,t_2;x,\xi)=\textstyle\sum_{j,k=0}^{\infty} t_1^j t_2^k P_{jk}(x,\xi)$$

defined in Ω is said to be equivalent to zero and is written $P(t_1,t_2;x,\xi)\sim 0$ if for any $\Omega'\subset\subset\Omega$ there exist constants d, A which satisfy the following conditions:

(a) $0<d$, $0<A<1$.

(b) For each $h>0$ there is a constant $C>0$ such that

$$\left|\sum_{j+k\leq m-1} P_{jk}(x,\xi)\right| \leq CA^m \exp(h|\xi|) \tag{2.3}$$

for any $m=1, 2, \cdots$; $(x, \xi) \in \Omega'$ satisfying $|\xi| \geq md$.

The set of all double formal symbols defined in Ω which are equivalent to zero is denoted by $\hat{R}_2(\Omega)$. We set $\hat{R}(\Omega)=\hat{R}_1(\Omega)=\hat{R}_2(\Omega)|_{t_2=0}$. We put furthermore $S(\Omega)=\hat{S}(\Omega)|_{t=0}$ and $R(\Omega)=\hat{R}(\Omega)\cap S(\Omega)$. Here we always consider $t=t_1$. Then there are the following injections:

$$\begin{array}{ccccc} R(\Omega) & \hookrightarrow & \hat{R}(\Omega) & \hookrightarrow & \hat{R}_2(\Omega) \\ \cap\downarrow & & \cap\downarrow & & \cap\downarrow \\ S(\Omega) & \hookrightarrow & \hat{S}(\Omega) & \hookrightarrow & \hat{S}_2(\Omega). \end{array}$$

It is easy to see that $\hat{R}_2(\Omega)$ (resp. $\hat{R}(\Omega)$, $R(\Omega)$) is an ideal of $\hat{S}_2(\Omega)$ (resp. $\hat{S}(\Omega)$, $S(\Omega)$) and that $\hat{S}(\Omega)\cap\hat{R}_2(\Omega)=\hat{R}(\Omega)$. Hence there is an injective homomorphism

$$\iota_{12}: \hat{S}(\Omega)/\hat{R}(\Omega) \longrightarrow \hat{S}_2(\Omega)/\hat{R}_2(\Omega)$$

induced from the preceding inclusions. On the other hand we can define a homomorphism

$$\rho_{21}: \hat{S}_2(\Omega)/\hat{R}_2(\Omega) \longrightarrow \hat{S}(\Omega)/\hat{R}(\Omega)$$

by setting $\rho_{21}(P(t_1, t_2; x, \xi))=P(t, t; x, \xi)$. Then we have $\rho_{21}\circ\iota_{12}=id$, $\iota_{12}\circ\rho_{21}=id$. By the theory of symbols of holomorphic microlocal operators (cf. [4]), $\varinjlim \hat{S}(\Omega)/\hat{R}(\Omega)$ ($\Omega \ni \mathring{x}^*$; conic neighborhood) is additively isomorphic to the stalk $\mathcal{E}^R_{\mathring{x}^*}$ of $\mathcal{E}^R_X$ at $\mathring{x}^*$. Therefore we have

Proposition 3. *There is an additive isomorphism*

$$\varinjlim \hat{S}_2(\Omega)/\hat{R}_2(\Omega) \longrightarrow \mathcal{E}^R_{\mathring{x}^*}$$

such that the image of $x_j\xi_j$ is equal to x_jD_j $(j=1, \cdots, n)$.

Definition 4. The image of a double formal symbol $P(t_1, t_2; x, \xi) = \sum_{j,k} t_1^j t_2^k P_{jk}(x, \xi)$ under the preceding isomorphism is denoted by $:P(t_1, t_2; x, \xi): = :\sum_{j,k} t_1^j t_2^k P_{jk}(x, \xi):$ and is said to be the normal product of $P(t_1, t_2; x, \xi)$. We often abbreviate $:\sum t_1^j t_2^k P_{jk}(x, \xi):$ to $:\sum P_{jk}(x, \xi):$.

Let $P(t; x, \xi)$, $Q(t; x, \xi)$ be formal symbols ($\in \hat{S}(\Omega)$). Then the composite operator $:P(t; x, \xi): \ :Q(t; x, \xi):$ is expressed in terms of double symbols as follows.

Proposition 5. *Set*

$$W(t_1, t_2; x, \xi) = \exp(t_2\partial_\xi\cdot\partial_y)P(t_1; x, \xi)Q(t_1; y, \eta)\Big|_{\substack{y=x\\ \eta=\xi}}. \tag{2.4}$$

Then $W(t_1, t_2; x, \xi)$ is a double formal symbol satisfying

$$:W(t_1, t_2; x, \xi): = :P(t; x, \xi): \ :Q(t; x, \xi):. \tag{2.5}$$

3. Statement of the results. A formal symbol $P(t; x, \xi) = \sum_{j=0}^\infty t^j P_j(x, \xi)$ defined in Ω is said to be of order at most m (m is a real number) if for any $\Omega' \Subset \Omega$ there are constants d, A satisfying the following conditions:

(a) $0<d$, $0<A<1$.

(b) There is a constant $C>0$ such that

$$|P_j(x, \xi)| \leq CA^j|\xi|^m$$

for any $j=0, 1, 2, \cdots$; $(x, \xi) \in \Omega'$, $|\xi| \geq (j+1)d$.

A formal symbol $p(t; x, \xi)$ is said to be of order at most $1-0$ if it satisfies the condition of Proposition 2 in [2].

Now let $p(t; x, \xi)$, $q(t; x, \xi)$ be formal symbols of order at most $1-0$ defined in Ω. Let $a(t; x, \xi)$ and $b(t; x, \xi)$ be formal symbols of order at most m_1 and m_2 respectively defined in Ω. We introduce two sequences $\{w_j\}$, $\{\psi_j\}$ of formal symbols defined in $\Omega \times \Omega$ as follows:

$$\tag{3.1} \begin{cases} w_0(t; x, y, \xi, \eta) = p(t; x, \xi) + q(t; y, \eta), \\ \psi_0(t; x, y, \xi, \eta) = a(t; x, \xi) \cdot b(t; y, \eta), \\ w_{j+1} = \dfrac{1}{j+1}\Big(\partial_\xi \cdot \partial_y w_j + \sum\limits_{\mu=0}^{j} \partial_\xi w_\mu \cdot \partial_y w_{j-\mu}\Big), \\ \psi_{j+1} = \dfrac{1}{j+1}\Big\{\partial_\xi \cdot \partial_y \psi_j + \sum\limits_{\mu=0}^{j} (\partial_\xi \psi_\mu \cdot \partial_y w_{j-\mu} + \partial_y \psi_\mu \cdot \partial_\xi w_{j-\mu})\Big\}. \end{cases}$$

Here $j=0, 1, 2, \cdots$. Let us consider formal series

$$r(t; x, \xi) = \sum_{j=0}^{\infty} t^j w_j(t; x, x, \xi, \xi),$$

$$c(t; x, \xi) = \sum_{j=0}^{\infty} t^j \psi_j(t; x, x, \xi, \xi).$$

Then we have

Theorem 6. *The formal series $r(t; x, \xi)$ and $c(t; x, \xi)$ are formal symbols of order at most $1-0$ and m_1+m_2 respectively defined in Ω satisfying*

$$\tag{3.2} \begin{aligned} &: a(t; x, \xi) \cdot \exp\{p(t; x, \xi)\} : \; : b(t; x, \xi) \cdot \exp\{q(t; x, \xi)\} : \\ &\quad = : c(t; x, \xi) \cdot \exp\{r(t; x, \xi)\} : . \end{aligned}$$

Remarks. (a) Of course such an expression as the right-hand side in (3.2) is not unique. We can, for example, replace c by $ce^{r'}$ for any r' to be of order at most 0 and r by $r-r'$.

(b) The preceding theorem is valid even for non-local operators so long as the right member makes sense. For instance, a kind of composition formula for Fourier integral operators (cf. [5]), or rather for "Laplace integral operators" (cf. [6]) is obtained.

When $a=b=1$, we have the following as a corollary of Theorem 6.

Theorem 7. *The formal series $r(t; x, \xi)$ is a formal symbol of order at most $1-0$ defined in Ω such that*

$$\tag{3.3} : \exp\{p(t; x, \xi)\} : \; : \exp\{q(t; x, \xi)\} : = : \exp\{r(t; x, \xi)\} : .$$

4. Outline of the proof of Theorem 6. The composite operator $: ae^p : \; : be^q :$ is expressed by Proposition 5. That is, if we set

$$\Pi = \exp(t_2 \partial_\xi \cdot \partial_y) a(t; x, \xi) b(t; y, \eta) \exp\{p(t; x, \xi) + q(t; y, \eta)\}$$

then we have $: ae^p : \; : be^q : = : \Pi|_{y=x, \eta=\xi} :$. The double formal symbol Π (defined in $\Omega \times \Omega$) is the unique solution of

$$\tag{4.1} \begin{cases} \partial_{t_2} \Pi = \partial_\xi \cdot \partial_y \Pi, \\ \Pi|_{t_2=0} = a(t; x, \xi) b(t; y, \eta) \exp\{p(t; x, \xi) + q(t; y, \eta)\}. \end{cases}$$

We assume Π to be of the form

$$\Pi=\sum_{j=0}^{\infty} t_2^j \psi_j \exp\left(\sum_{k=0}^{\infty} t_2^k w_k\right).$$

If $\{\psi_j\}$ and $\{w_k\}$ satisfy (3.1), then Π is a solution to (4.1). Moreover one can see that $\sum t_2^j \psi_j$ and $\sum t_2^k w_k$ themselves are double formal symbols of order at most m_1+m_2 and $1-0$ respectively defined in $\Omega\times\Omega$. Since

$$c(t;x,\xi)\sim\sum t_2^j \psi_j(t;x,x,\xi,\xi),$$
$$r(t;x,\xi)\sim\sum t_2^j w_j(t;x,x,\xi,\xi),$$

we obtain the theorem.

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