143. *Class Numbers of Positive Definite Ternary Quaternion Hermitian Forms**)

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0. In the previous papers [2], [3], we have studied the class numbers of positive definite quaternary hermitian forms; there we have classified the conjugacy classes of the group of similitudes of our forms for arbitrary rank n, and worked out explicit formulas for the class numbers of genera of maximal lattices, in the binary case ($n=2$), by using author's general formula for the traces of Brandt matrices associated with such forms ([1]). The purpose of this note is to announce a similar result in the ternary case ($n=3$), under the condition that the discriminant is a prime p. The general case, as well as the proofs, being somewhat lengthy, will appear elsewhere.

1. Let B denote a definite quaternion algebra over $\boldsymbol{Q}$ and $V=B^n$ be a left B-space of rank n. We regard V as a positive hermitian space over B by the metric $F(x)=\sum_{i=1}^n \mathrm{Nr}(x_i)$, $x=(x_i)\in V$, where $\mathrm{Nr}(a)=a\bar{a}$ denotes the reduced norm of B. Then the group $G=G(V,F)$ of all similitudes of (V,F) is given by

$$G=\{g\in M_n(B)\,;\, g\,{}^t\bar{g}=n(g)\cdot 1_n,\ n(g)\in \boldsymbol{Q}^\times\}.$$

Let $\mathcal{O}$ be a maximal order of B. We regard $\mathcal{O}^n$ as a lattice in V and denote by $\mathcal{L}(\mathcal{O})$ the G-genus of $\mathcal{O}$-lattices containing $\mathcal{O}^n$. It is called *the principal genus*. By definition, an $\mathcal{O}$-lattice L in V belongs to $\mathcal{L}(\mathcal{O})$ if and only if $L_p=(\mathcal{O}_p^n)g_p$, $g_p\in G_p$ for all prime p, where $\mathcal{O}_p$, L_p, and G_p are p-adic completions of $\mathcal{O}$, L, and G respectively. The adelized group G_A of G acts transitively on $\mathcal{L}(\mathcal{O})$ by $Lg=\bigcap_p (L_pg_p\cap V)$, and the stabilizer of $\mathcal{O}^n$ in G_A is given by $\mathfrak{U}=G_{\boldsymbol{R}}\times\prod_p U_p$, $U_p=G_p\cap GL_n(\mathcal{O}_p)$. The number H of the classes (i.e., G-orbits) in $\mathcal{L}(\mathcal{O})$ is then equal to the number of $(\mathfrak{U}, G)$-double cosets in $\mathfrak{U}\backslash G_A/G$.

2. By [1], Theorem 1, the class number H, being equal to the trace of the Brandt matrix $B_\rho(1)$ with $\rho=1$, is expressed as follows

$$(*)\qquad H=\mathrm{tr}\,B_1(1)=\sum_{C(g)}\sum_{L_G(\Lambda)} M_G(\Lambda)\prod_p c_p(g,U_p,\Lambda_p),$$

where the notations are as follows: put, for each element g of G,

$$Z(g)=\{z\in M_n(B)\,;\, zg=gz\},\qquad Z_G(g)=Z(g)^\times\cap G.$$

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i) $C(g)$ runs over the conjugacy classes in G represented by g, which satisfies (1) $n(g)=1$, (2) $C(g)$ is locally integral i.e., $C_A(g)\cap \mathfrak{U}\neq\phi$ (Note that (1), (2) imply that g is of finite order).

ii) $L_G(\Lambda)$ runs over the set of G-genera of Z-orders of $Z(g)$.

iii) $M_G(\Lambda)$ is the G-$Ma\beta$ (or G-measure) of $L_G(\Lambda)$.

iv) $c_p(g, U_p, \Lambda_p)=\sharp(Z_G(g)_p\backslash M_p(g, U_p, \Lambda_p)/U_p)$,

$$M_p(g, U_p, \Lambda_p)=\left\{x_p\in G_p\,;\ \begin{matrix} x_pgx_p^{-1}\in U_p, \text{ and } Z(g)_p\cap x_pM_n(\mathcal{O}_p)x_p^{-1} \\ =a_p^{-1}\Lambda_pa_p \text{ for some } a_p\in Z_G(g)_p \end{matrix}\right\}.$$

We refer to [1], for more precise definitions. In general, to work out from (*) the explicit formula for H is not easy; it requires the classification of conjugacy classes in G, G_p, and U_p, and computation of $M_G(\Lambda)$, which are carried out with long calculations.

3. Throughout the following, we assume that $n=3$. Then the principal polynomials of torsion elements of G which take parts in the formula (*) are:

$f_1(x)=(x-1)^6$, $f_2(x)=(x-1)^4(x+1)^2$ $f_3(x)=(x-1)^4(x^2+1)$,
$f_4(x)=(x-1)^4(x^2+x+1)$, $f_5(x)=(x-1)^4(x^2-x+1)$,
$f_6(x)=(x-1)^2(x+1)^2(x^2+1)$, $f_7(x)=(x-1)^2(x+1)^2(x^2+x+1)$,
$f_8(x)=(x^2+1)^3$, $f_9(x)=(x^2+x+1)^3$, $f_{10}(x)=(x-1)^2(x^2+1)^2$,
$f_{11}(x)=(x-1)^2(x^2+x+1)^2$, $f_{12}(x)=(x-1)^2(x^2-x+1)^2$,
$f_{13}(x)=(x^2+x+1)(x^2+1)^2$, $f_{14}(x)=(x^2+1)(x^2+x+1)^2$,
$f_{15}(x)=(x^2+x+1)(x^2-x+1)^2$, $f_{16}(x)=(x-1)^2(x^2+1)(x^2+x+1)$,
$f_{17}(x)=(x-1)^2(x^2+1)(x^2-x+1)$, $f_{18}(x)=(x-1)^2(x^2+x+1)(x^2-x+1)$,
$f_{19}(x)=(x^2+1)(x^2+x+1)(x^2-x+1)$, $f_{20}(x)=(x-1)^2(x^4+x^3+x^2+x+1)$,
$f_{21}(x)=(x-1)^2(x^4-x^3+x^2-x+1)$, $f_{22}(x)=(x-1)^2(x^4+1)$,
$f_{23}(x)=(x-1)^2(x^4-x^2+1)$, $f_{24}(x)=(x^2+1)(x^4+x^3+x^2+x+1)$,
$f_{25}(x)=(x^2+1)(x^4+1)$, $f_{26}(x)=(x^2+1)(x^4-x^2+1)$,
$f_{27}(x)=(x^2+x+1)(x^4+x^3+x^2+x+1)$,
$f_{28}(x)=(x^2+x+1)(x^4-x^3+x^2+x+1)$, $f_{29}(x)=(x^2+x+1)(x^4+1)$,
$f_{30}(x)=(x^2+x+1)(x^4-x^2+1)$, $f_{31}(x)=(x^6+x^5+x^4+x^3+x^?+x+1)$,
$f_{32}(x)=(x^6+x^3+1)$,

and $f_i(\pm x)$ $(1\leqq i\leqq 32)$.

4. We denote by H_i $(1\leqq i\leqq 32)$ the total contribution to the formula (*) of those elements whose principal polynomials are $f_i(\pm x)$. Then our result in the case where B has the prime discriminant p is stated in the following

Theorem. *Under the assumption that B has the prime discriminant p, the class number H of the principal genus $\mathcal{L}(\mathcal{O})$ in the positive definite ternary space (V, F) is given by*

$$H=\sum_{i=1}^{32}H_i,$$
$$H_1=2^{-9}3^{-4}5^{-1}7^{-1}(p-1)(p^2+1)(p^3-1),$$
$$H_2=31.2^{-9}3^{-3}5^{-1}(p-1)^2(p^2+1),$$

$$H_3=2^{-8}3^{-2}5^{-1}(p-1)(p^2+1)\left(1-\left(\frac{-1}{p}\right)\right),$$

$$H_4=H_5=2^{-7}3^{-3}5^{-1}(p-1)(p^2+1)\left(1-\left(\frac{-3}{p}\right)\right),$$

$$H_6=7.2^{-8}3^{-2}(p-1)^2\left(1-\left(\frac{-1}{p}\right)\right),$$

$$H_7=7.2^{-6}3^{-3}(p-1)^2\left(1-\left(\frac{-3}{p}\right)\right),$$

$$H_8=2^{-7}3^{-1}(p^2-p+2)\left(1-\left(\frac{-1}{p}\right)\right),$$

$$H_9=2^{-3}3^{-4}(p^2-p+2)\left(1-\left(\frac{-3}{p}\right)\right),$$

$$H_{10}=2^{-7}3^{-2}(p-1)\left[23(p-1)+9\left(1-\left(\frac{-1}{p}\right)\right)\right],$$

$$H_{11}=2^{-6}3^{-3}(p-1)\left[52(p-1)+2\left(1-\left(\frac{-3}{p}\right)\right)\right],$$

$$H_{12}=2^{-6}3^{-3}(p-1)\left[4(p-1)+2\left(1-\left(\frac{-3}{p}\right)\right)\right],$$

$$H_{13}=2^{-5}3^{-2}\left(1-\left(\frac{-3}{p}\right)\right)\left[5(p-1)+3\left(1-\left(\frac{-1}{p}\right)\right)\right],$$

$$H_{14}=2^{-5}3^{-2}\left(1-\left(\frac{-1}{p}\right)\right)\left[4(p-1)+2\left(1-\left(\frac{-3}{p}\right)\right)\right],$$

$$H_{15}=2^{-3}3^{-3}\left(1-\left(\frac{-3}{p}\right)\right)\left[5(p-1)+7\left(1-\left(\frac{-3}{p}\right)\right)\right],$$

$$H_{16}=H_{17}=2^{-5}3^{-2}(p-1)\left(1-\left(\frac{-1}{p}\right)\right)\left(1-\left(\frac{-3}{p}\right)\right),$$

$$H_{18}=2^{-2}3^{-3}(p-1)\left(1-\left(\frac{-3}{p}\right)\right)^2,$$

$$H_{19}=2^{-2}3^{-2}\left(1-\left(\frac{-1}{p}\right)\right)\left(1-\left(\frac{-3}{p}\right)\right)^2,$$

$$H_{20}=H_{21}=2^{-3}3^{-1}5^{-1}(p-1)[1,0,0,0,4;5],$$

$$H_{22}=2^{-4}3^{-1}(p-1)[*,0,*,1,*,1,*,2;8],$$

$$H_{23}=2^{-3}3^{-2}(p-1)[*,0,*,*,*,1,*,0,*,*,*,1;12],$$

$$H_{24}=2^{-1}5^{-1}\left(1-\left(\frac{-1}{p}\right)\right)[1,0,0,0,2;5],$$

$$H_{25}=3.2^{-3}[*,0,*,1,*,0,*,2;8],$$

$$H_{26}=2^{-2}3^{-1}[*,0,*,*,*,0,*,4,*,*,*,5;12],$$

$$H_{27}=H_{28}=2^{-1}3^{-1}5^{-1}\left(1-\left(\frac{-3}{p}\right)\right)[1,0,0,0,4;5],$$

$$H_{29}=2^{-2}3^{-1}\left(1-\left(\frac{-3}{p}\right)\right)[*, 0, *, 1, *, 1, *, 2\,;\, 8],$$

$$H_{30}=3^{-2}[*, 0, *, *, *, 1, *, 0, *, *, *, 4\,;\, 12],$$

$$H_{31}=7^{-1}[1, 0, 0, 2, 0, 2, 8\,;\, 7],$$

$$H_{32}=3^{-2}[*, 0, 2, *, 0, 2, *, 0, 8\,;\, 9],$$

where $t=t(p)=[t_0, t_1, \cdots, t_{q-1}\,;\, q]$ *means that* $t=t_j$ *if* $p\equiv j \pmod q$.

Example.

p	2	3	5	7	11	13	17	19	23	29	31	37	41
H	1	2	3	5	19	23	70	109	262	755	1047	2586	4526

References

[1] K. Hashimoto: On Brandt matrices associated with the positive definite quaternion hermitian forms. J. Fac. Sci. Univ. of Tokyo, **27**, 227–245 (1980).

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[3] ——: ditto. II. ibid., **28**, 695–699 (1982).