141. *On Certain Integrals over Spheres*

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§ 1. Statement of the formula (F). Let us begin with a well-known formula of the complete elliptic integral

$$(1.1)\qquad \frac{2}{\pi}K \overset{\text{def}}{=\!=} \frac{2}{\pi}\int_0^{\pi/2}(1-\lambda\sin^2\theta)^{-1/2}d\theta = {}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;\lambda\Big),$$

where $\lambda\in\boldsymbol{C}$, $|\lambda|<1$ and ${}_2F_1$ is the Gauss' hypergeometric series. If we pass to the cartesian coordinates $x=\cos\theta$, $y=\sin\theta$ of the plane $\boldsymbol{R}^2$, then (1.1) becomes

$$(1.2)\qquad \int_{S^1}(1-\lambda y^2)^{-1/2}d\omega = {}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;\lambda\Big)$$

where, in general, S^{n-1} denotes the unit sphere of $\boldsymbol{R}^n$ with the center at the origin and $d\omega$ is the volume element of S^{n-1} such that the volume of S^{n-1} is 1. Notice here that y^2 is a degenerate quadratic form on $\boldsymbol{R}^2$.

In this note, we shall give a generalization of (1.2). Namely, along with a partition $n=p+q$, p, $q>0$, of an integer n, consider the decomposition $\boldsymbol{R}^n=\boldsymbol{R}^p\oplus\boldsymbol{R}^q$ of the euclidean space $\boldsymbol{R}^n$. When $z=(x,y)\in\boldsymbol{R}^n$, $x\in\boldsymbol{R}^p$, $y\in\boldsymbol{R}^q$, we have $Nz=Nx+Ny$, where $Nx=x_1^2+\cdots+x_p^2$, etc. Let a, b be non-negative integers such that $c=a+b>0$. Our generalization of (1.2) is the following formula:

$$(\mathrm{F})\qquad \int_{S^{n-1}}(1-\lambda(Nx)^a(Ny)^b)^{-s}d\omega = {}_{c+1}F_c(s,\alpha;\beta;a^ab^bc^{-c}\lambda),$$

where $\lambda\in\boldsymbol{C}$, $|\lambda|<1$, $s\in\boldsymbol{C}$ and

$$\alpha=\Big(\frac{p}{2a},\frac{p+2}{2a},\cdots,\frac{p+2(a-1)}{2a},\frac{q}{2b},\frac{q+2}{2b},\cdots,\frac{q+2(b-1)}{2b}\Big),$$

$$\beta=\Big(\frac{n}{2c},\frac{n+2}{2c},\cdots,\frac{n+2(c-1)}{2c}\Big).$$

Needless to say, (1.2) is a special case of (F) where $n=2$, $p=q=1$, $a=0$, $b=c=1$ and $s=1/2$. We remind the reader the definition of the (generalized) hypergeometric series which appears on the right hand side of (F). First, for $a\in\boldsymbol{C}$, $k\in\boldsymbol{Z}$, $k\geqq 0$, we put, following Appell,

$$(a,k)=\begin{cases} a(a+1)\cdots(a+k-1), & k>0,\\ 1, & k=0.\end{cases}$$

Next, for integers μ, $\nu\geqq 0$, consider vectors $\alpha=(\alpha_1,\cdots,\alpha_\mu)\in\boldsymbol{C}^\mu$, $\beta=(\beta_1,\cdots,\beta_\nu)\in\boldsymbol{C}^\nu$. The hypergeometric series ${}_\mu F_\nu(\alpha;\beta;z)$, $z\in\boldsymbol{C}$, is then defined by

$$ {}_{\mu}F_{\nu}(\alpha\,;\beta\,;z)=\sum_{k=0}^{\infty}\frac{(\alpha_1,k)\cdots(\alpha_\mu,k)z^k}{(\beta_1,k)\cdots(\beta_\nu,k)k\,!}. $$

§2. Proof of the formula (F). Since both sides of (F) are holomorphic for $\lambda\in\boldsymbol{C}$, $|\lambda|<1$, we may assume that $\lambda\in\boldsymbol{R}$, $|\lambda|<1$. Let us put

$$ g(z)=1-\lambda(Nx)^a(Ny)^b,\qquad z=(x,y)\in\boldsymbol{R}^n=\boldsymbol{R}^p\oplus\boldsymbol{R}^q, \tag{2.1} $$

$$ \theta(t)=\int_{S^{n-1}}e^{-tg(z)}d\omega,\qquad t>0, \tag{2.2} $$

and

$$ \varphi(s)=\int_0^\infty t^{s-1}\theta(t)dt,\qquad s\in\boldsymbol{C}. \tag{2.3} $$

Since $g(z)>0$ on S^{n-1}, it is easy to see that $\varphi(s)$ is holomorphic for $\mathrm{Re}(s)>0$ as in the case of the Euler integral for the gamma function. We shall look at $\varphi(s)$ in two ways. First, by the change of variable $u=tg(z)$, we have

$$ \varphi(s)=\Gamma(s)\int_{S^{n-1}}g(z)^{-s}d\omega \tag{2.4} $$

where the integral represents an entire function of s since $g(z)>0$ on S^{n-1}. Next, we start with

$$ \varphi(s)=\int_0^\infty t^{s-1}e^{-t}dt\int_{S^{n-1}}e^{\lambda(Nx)^a(Ny)^b}d\omega. \tag{2.5} $$

For the moment, we shall assume the following equality

$$ \int_{S^{n-1}}e^{u(Nx)^a(Ny)^b}d\omega={}_cF_c(\alpha\,;\beta\,;a^ab^bc^{-c}u),\qquad u\in\boldsymbol{C}, \tag{2.6} $$

proof of which will be given soon. Substituting (2.6) in (2.5) with $u=\lambda t$ and using Mellin's formula for hypergeometric series,[1] we get

$$ \begin{aligned}\varphi(s)&=\int_0^\infty t^{s-1}e^{-t}\,{}_cF_c(\alpha\,;\beta\,;a^ab^bc^{-c}\lambda t)dt\\&=\Gamma(s)\,{}_{c+1}F_c(s,\alpha\,;\beta\,;a^ab^bc^{-c}\lambda).\end{aligned} \tag{2.7} $$

Then, our formula (F) follows from (2.4) and (2.7) on eliminating $\Gamma(s)$. We now concentrate on proving (2.6). Since $Nz=Nx+Ny=1$ on S^{n-1}, we have

$$ \begin{aligned}I&\overset{\text{def}}{=\!=}\int_{S^{n-1}}e^{u(Nx)^a(Ny)^b}d\omega=\sum_{k=0}^\infty\frac{u^k}{k\,!}\int_{S^{n-1}}(Nx)^{ak}(1-Nx)^{bk}\,d\omega\\&=\sum_{k=0}^\infty\frac{u^k}{k\,!}\sum_{\nu=0}^\infty\frac{(-bk,\nu)}{\nu\,!}\int_{S^{n-1}}(Nx)^{ak+\nu}d\omega.\end{aligned} \tag{2.8} $$

As $\xi=(\underbrace{1,\cdots,1}_{p},\underbrace{0,\cdots,0}_{q})$ is the set of eigenvalues of the degenerate quadratic form Nx on $\boldsymbol{R}^n$, we have

$$ \int_{S^{n-1}}(Nx)^{ax+\nu}\,d\omega=\frac{(ak+\nu)\,!}{4^{ak+\nu}(n/2,ak+\nu)}b_{ak+\nu}(2\,;\xi) \tag{2.9} $$

where the numbers $b_{ak+\nu}(2\,;\xi)$ are determined by the generating relation

1) See [1] p. 63, line 10 from the bottom.

$$\sum_{\mu=0}^{\infty} b_\mu(2\,;\xi)t^\mu=\prod_{i=1}^{n}(1-4\xi_i t)^{-1/2}=(1-4t)^{-p/2}.\text{[2]} \tag{2.10}$$

Therefore, we have

$$b_\mu(2\,;\xi)=\frac{(p/2,\mu)4^\mu}{\mu\,!}. \tag{2.11}$$

From (2.9), (2.11), we get

$$I=\sum_{k=0}^{\infty}\frac{u^k}{k\,!}\frac{(p/2,ak)}{(n/2,ak)}{}_2F_1\left(-bk,\frac{p}{2}+ak\,;\frac{n}{2}+ak\,;1\right). \tag{2.12}$$

As is well-known, we have

$${}_2F_1(\alpha,\beta\,;\gamma\,;1)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \tag{2.13}$$

when $\mathrm{Re}(\gamma-\alpha-\beta)>0$ and γ is not a non-positive integer.[3] Our formula (2.6) follows at once form (2.12), (2.13) and the following multiplication formula for Appell's symbol

$$(\alpha,mk)=m^{mk}\left(\frac{\alpha}{m},k\right)\left(\frac{\alpha+1}{m},k\right)\cdots\left(\frac{\alpha+m-1}{m},k\right).$$

§3. Application of the formula (F) to deformations of Hopf maps. Let Ω be an open set of $\boldsymbol{R}^n$ containing S^{n-1} and $f:\Omega\to\boldsymbol{R}^m$ be a continuous map. Assume further that $f(z)\neq 0$ for all $z\in S^{n-1}$. Then we obtain an entire function of $s\in\boldsymbol{C}$ defined by

$$K(f\,;s)=\int_{S^{n-1}}N(f(z))^{-s}d\omega. \tag{3.1}$$

If $N(f(z))$ happens to be of the form $1-\lambda(Nx)^a(Ny)^b$ for a suitable decomposition $\boldsymbol{R}^n=\boldsymbol{R}^p\oplus\boldsymbol{R}^q$ and integers a, b as in §1, then, by the formula (F), $K(f\,;s)$ may be expressed as a hypergeometric series. This is the case of the deformation of Hopf maps. Namely, suppose that there is a bilinear map $B:\boldsymbol{R}^p\times\boldsymbol{R}^q\to\boldsymbol{R}^r$ such that

$$N(B(x,y))=NxNy,\quad x\in\boldsymbol{R}^p,\quad y\in\boldsymbol{R}^q.$$

For $\lambda\in\boldsymbol{R}$ such that $|\lambda|<1/4$, put

$$f_\lambda(z)=(Nx-Ny,\ 2(1-\lambda)^{1/2}B(x,y)). \tag{3.2}$$

Then, we have

$$N(f_\lambda(z))=1-4\lambda(Nx)(Ny). \tag{3.3}$$

For $\lambda=0$, f_0 is a Hopf map. With $a=b=1$, $c=2$, we get from the formula (F)

$$K(f_\lambda\,;s)={}_3F_2\left(s,\frac{p}{2},\frac{q}{2}\,;\frac{n}{4},\frac{n+2}{4}\,;\lambda\right). \tag{3.4}$$

If, in particular, $p=q=r=n/2$, then, by a theorem of Hurwitz, only 4 cases: $p=1, 2, 4, 8$, are possible. These cases are materialized by the classical Hopf fibration: $S^1\to S^1$, $S^3\to S^2$, $S^7\to S^4$ and $S^{15}\to S^8$.

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2) See §1, §3 of [3].

3) See [2] p. 49, Th. 18.

Since $q=n/2$, the series ${}_3F_2$ in (3.4) degenerates to the Gauss' series ${}_2F_1$:

$$K(f_\lambda; s) = {}_2F_1\left(s, \frac{p}{2}; \frac{p+1}{2}; \lambda\right).^{4)} \tag{3.5}$$

If we use the Euler integral representation of Gauss' series, we obtain

$$\begin{aligned} K(f_\lambda; s) &= \frac{(p-1)!}{2^{p-1}\Gamma(p/2)^2}\int_0^1 z^{p/2-1}(1-z)^{-1/2}(1-\lambda z)^{-s}dz \\ &= \frac{2(p-1)!}{2^{p-1}\Gamma(p/2)^2}\int_0^{\pi/2} \sin^{p-1}\theta(1-\lambda\sin^2\theta)^{-s}d\theta. \end{aligned} \tag{3.6}$$

If, in particular, $p=1$ and $s=1/2$, then f_λ are deformations of the double covering $S^1 \to S^1$ given by the squaring $z \mapsto z^2$, $z \in \boldsymbol{C}$, and (3.6) boils down to the complete elliptic integral $(2/\pi)K$ in (1.1).

References

[1] W. Magnus, F. Oberhettinger, and R. P. Soni: Formulas and Theorems for the Special Functions of Mathematical Physics. 3rd ed., Springer-Verlag, New York (1966).

[2] E. D. Rainville: Special Functions. Macmillan, New York (1960).

[3] T. Ono: On a generalization of Laplace integrals (to appear in Nagoya Math. J., vol. 92 (1983)).

[4] ——: On deformations of Hopf maps and hypergeometric series (manuscript).

[5] ——: A generalization of Gauss' theorem on arithmetic-geometric means. Proc. Japan Acad., **59**A, 154–157 (1983).

4) When s is a negative integer, the formula (3.5) was obtained by a different method in [4].