140. *Fundamental Theorems in Global Knot Theory. II*

By Itiro TAMURA
Department of Mathematics, Faculty of Science, University of Tokyo

(Communicated by Shokichi IYANAGA, M. J. A., Dec. 12, 1983)

1. Genus one knot in $S^n \times S^{n+1}$. We specify orientations of the n-sphere S^n and the $(n+1)$-sphere S^{n+1} and give $S^n \times S^{n+1}$ the product orientation. Let S_0^n denote the equator of S^{n+1} and let us consider submanifolds $S^n \times S_0^n$ and $\{z_0\} \times S^{n+1}$ of $S^n \times S^{n+1}$, where $z_0 \in S^n$.

Let $f: S^n \to S^n \times S_0^n$ be an imbedding having the following properties:

(i) The degree of $p_1 \circ f: S^n \to S^n$ is m, where $p_1: S^n \times S_0^n \to S^n$ is the projection onto the first factor.

(ii) $f(S^n)$ and $\{z_0\} \times S^{n+1}$ intersect transversally at finite points (z_0, u_i) $(i=0, 1, 2, \cdots, s)$, (z_0, v_i) $(i=1, 2, \cdots, t)$ so that the intersection number is 1 at (z_0, u_i) and -1 at (z_0, v_i).

(iii) the normal bundle of $f(S^n)$ in $S^n \times S_0^n$ is trivial.

Let $g: S^n \to \{z_0\} \times S^{n+1}$ be an imbedding having the following properties:

(i) A neighborhood $U(z_0, u_0)$ of (z_0, u_0) in $\{z_0\} \times S_0^n$ is contained in $g(S^n)$.

(ii) Let $\hat{D}$ and $\hat{D}'$ denote connected components of $\{z_0\} \times S^{n+1} - g(S^n)$. Then the points (z_0, u_i) $(i=1, 2, \cdots, s')$ and (z_0, v_i) $(i=1, 2, \cdots, t')$ are contained in $\hat{D}$, and the points (z_0, u_i) $(i=s'+1, s'+2, \cdots, s)$ and (z_0, v_i) $(i=t'+1, t'+2, \cdots, t)$ are contained in $\hat{D}'$, where $1 \leqq s' \leqq s$, $1 \leqq t' \leqq t$ and $q=s'-t'$.

Let $N(f)$ be a tubular neighborhood of $f(S^n)$ in $S^n \times S_0^n$ and let $N(g) = D^n(z_0) \times g(S^n)$, where $D^n(z_0)$ is an n-disk with center z_0 in S^n. We can choose $N(f)$ and $N(g)$ so that $N(f) \cap N(g)$ is diffeomorphic to the $2n$-disk.

Let $V_{m,q} = N(f) \cup N(g)$ be the plumbing of $N(f)$ and $N(g)$. $V_{m,q}$ is called a *genus one Seifert surface of type* (m, q). The boundary $\partial V_{m,q}$ of $V_{m,q}$ is a knot in $S^n \times S^{n+1}$ which is called a *genus one knot of type* (m, q) in $S^n \times S^{n+1}$ and is denoted by $K_{m,q}$. By proposition 2 of [2], the knot $K_{m,q}$ is simple.

According to the localness and unknotting theorem [2, Theorems 3, 4], we have the following theorem by computing the knot modules $A_i(K; S^n \times S^{n+1})$.

Theorem 1. *Let $K_{m,q}$ be a genus one knot of type (m, q) in $S^n \times S^{n+1} (n \geqq 3)$. Then the following hold:*

(i) *$K_{0,q}$ is local. And $K_{0,q}$ is unknotted if and only if $q=0$ or -1.*

(ii) *In case $m\neq 0$, $K_{m,q}$ is local if and only if m and q satisfy the following condition* (*):

(*) *Each prime factor of m divides q or $q+1$.*

Furthermore, $K_{m,q}$ ($m\neq 0$) is unknotted if it is local.

(iii) *$K_{m,q}$ are not fibred knots.*

The knot $K_{0,q}$ is essentially a knot in S^{2n+1}. The results of (i) and (ii) reveal the contrastive property of the knot theory in S^{2n+1} and the knot theory in $S^n\times S^{n+1}$.

Corollary 1. *A genus one knot $K_{m,q}$ of type (m,q) is inessential and not local in $S^n\times S^{n+1}$ if $m\neq 0$, and m and q do not satisfy the condition* (*) *in Theorem* 1.

2. Knot cobordisms. Two knots K_0 and K_1 in an m-dimensional smooth manifold M^m are said to be *cobordant* if there exists an $(m-1)$-dimensional submanifold W of $M^m\times[0,1]$ which satisfies the following conditions:

(i) W is diffeomorphic to $S^{m-2}\times I$.

(ii) $\partial W=W\cap((M^m\times\{0\})\cup(M^m\times\{1\}))$
$=(K_0\times\{0\})\cup(K_1\times\{1\})$.

A knot K cobordant to the trivial knot in M^m is said to be *null cobordant*.

It is obvious that homotopy classes in M^m represented by cobordant knots are same. In particular a null cobordant knot is inessential.

The cobordance is an equivalence relation in the set of knots in M^m. The set of the equivalence classes is called the *knot cobordism* in M^m and is denoted by $C_{m-2}(M^m)$. In case M^m is the m-sphere, $C_{m-2}(S^m)$ is simply denoted by C_{m-2}. By introducing orientations on knots and S^m, the knot cobordism C_{m-2} admits an abelian group structure by the connected sum.

Kervaire proved that $C_{2n}=0$ ($n\geqq 2$) ([1]). By similar method we can prove the following theorem.

Theorem 2. $C_{2n}(S^{n+1}\times S^{n+1})=0$ $(n\geqq 2)$.

The following corollary is a direct consequence of Theorem 2.

Corollary 2. *An element of the homotopy group $\pi_{2n}(S^{n+1}\times S^{n+1})$ $(n\geqq 2)$ is realizable by an imbedded $2n$-sphere in $S^{n+1}\times S^{n+1}$ if and only if it is the zero element.*

This result can be seen as a higher dimensional analogue of the problem of realization of 2-dimensional homotopy (homology) classes of $S^2\times S^2$ by imbedded 2-spheres from the viewpoint of codimension 2.

Details and proofs will appear elsewhere.

Added in proof: To the assumption (b) in Theorem 3 (Localness theorem) (I) and Theorem 4 (Unknotting theorem) (I) in the previous paper (Tamura [2]), the following condition should be added:

The $b \times b$ *matrix* $(I(\omega_i, \omega_j))$ *is unimodular.*

References

[1] M. Kervaire: Les noeuds de dimensions supérieurs. Bull. Soc. Math. France, **93**, 225–271 (1965).
[2] I. Tamura: Fundamental theorems in global knot theory. I. Proc. Japan Acad., **59**A, 446–448 (1983).