118. On Rational Similarity Solutions of KdV and m-KdV Equations

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§ 1. **Summary.** The Korteweg-de Vries (KdV) equation

$$u_t-12uu_x+u_{xxx}=0 \tag{1.1}$$

and the modified Korteweg-de Vries (m-KdV) equation

$$v_t-6v^2v_x+v_{xxx}=0 \tag{1.2}$$

have a series of rational similarity solutions

$$u_n(x,t)=g(n+1)x^{-2}-\left(\frac{\partial}{\partial x}\right)^2 \log F_{n+1}(x,t), \tag{1.3}$$

$$v_n(x,t)=(g(n)-g(n+1))x^{-1}+\frac{\partial}{\partial x}\log(F_n(x,t)/F_{n+1}(x,t)) \tag{1.4}$$

where

$$F_n(x,t)=\sum_{j=0}^{f(n)} P_{n,j}(3t)^j x^{3(f(n)-j)} \tag{1.5}$$

is a homogeneous polynomial of x^3 and t of degree $f(n)=[n(n-1)/6]$ with integral coefficients $P_{n,j}$ ($P_{n,0}=1$, $P_{n,f(n)}\neq 0$), $g(n)=1$ if $n\equiv 2 \pmod 3$, $=0$ otherwise. These polynomials are essentially the same as those of A. I. Yablonskii [1] and A. P. Vorobiev [2]. Actually the polynomials

$$P_n(\xi)=\sum_{j=0}^{f(n)} P_{n,j}\xi^{d(n)-3j}, \qquad (d(n)=n(n-1)/2) \tag{1.6}$$

were introduced by them to describe the rational solutions of Painlevé-II equation.

$$q_n=(\log P_n(\xi)/P_{n+1}(\xi))' \tag{1.7}$$

satisfies Painlevé-II equation

$$q_n''=2q_n^3+\xi q_n+n. \tag{1.8}$$

It gives also a rational solution of the Toda equation. If p_n is given by

$$p_n=-P_nP_{n+2}/4P_{n+1}^2=(\log P_{n+1}(\xi))''-\xi/4 \tag{1.9}$$

then $\{q_n, p_n\}$ satisfies the Toda equation

$$q_n'=p_{n-1}-p_n, \qquad p_n'=p_n(q_n-q_{n+1}). \tag{1.10}$$

Vorobiev calculated the coefficients of P_n ($n\leq 8$) and showed that $P_{n,j}$ are very large integers for large n and j. Here we give a theoretical bound for them.

$$|P_{n,j}|\leq (7n)^{4j}, \quad n=1,2,3,\cdots, \quad 0\leq j\leq f(n). \tag{1.11}$$

The zeros $a_{n,k}$ of P_n are all simple. P_n and P_{n+1} have no common zero. Using these zeros of P_n, u_n and v_n can be expressed as

$$u_n=\sum_{k=1}^{d(n+1)}(x-a_{n+1,k}(3t)^{1/3})^{-2}, \tag{1.12}$$

$$v_n=\sum_{k=1}^{d(n)}(x-a_{n,k}(3t)^{1/3})^{-1}-\sum_{k=1}^{d(n+1)}(x-a_{n+1,k}(3t)^{1/3})^{-1}. \tag{1.13}$$

So u_n has $n(n+1)/2$ double poles, v_n has n^2 simple poles on the complex x-plane.

In the previous work [3] we gave a sharp estimate for the maximal modulus $A_n=\max_{1\le k\le d(n)}|a_{n,k}|$ of these poles.

$$n^{2/3}\le A_{n+2}\le 4n^{2/3} \qquad n=0,1,2,\cdots \tag{1.14}$$

u_n and v_n have no singularities in $|x|^3>3A_{n+1}^3|t|$. They satisfy singular initial conditions

$$u_n(x,0)=d(n+1)x^{-2}, \qquad v_n(x,0)=-nx^{-1}. \tag{1.15}$$

M. J. Ablowitz and J. Satsuma [4] obtained these rational solutions u_n of KdV equation by Bäcklund transformation and showed that they can be obtained as a limit of anti soliton solutions.

§ 2. Recurrence relation. The rational functions v_n and u_n are uniquely determined by the recurrence relation

$$v_0=u_0=0, \tag{2.1}$$

$$v_n=-v_{n-1}-(2n-1)/(12tu_{n-1}+x), \tag{2.2}$$

$$u_n=v_n^2-u_{n-1}, \tag{2.3}$$

$$v_{-n}=-v_n, \quad u_{-n}=u_{n-1}, \quad n=1,2,3,\cdots. \tag{2.4}$$

Theorem 2.1. *$u_n(v_n)$ satisfies* KdV (m-KdV) *equation* (1.1) ((1.2)).

To show this it is convenient to introduce rational functions q_n and p_n by

$$q_n(\xi)=v_n(\xi,1/3), \quad \tilde{p}_n(\xi)=u_n(\xi,1/3), \quad p_n(\xi)=-\tilde{p}_n(\xi)-\xi/4. \tag{2.5}$$

q_n and p_n satisfy the following relations.

$$q_0=0, \qquad p_0=-\xi/4, \tag{2.6}$$

$$q_n=(2n-1)/4p_{n-1}-q_{n-1}, \qquad p_n=-p_{n-1}-q_n^2-\xi/2, \tag{2.7}$$

$$q_{-n}=-q_n, \quad p_{-n}=p_{n-1}, \quad n=1,2,3,\cdots. \tag{2.8}$$

As is shown in the previous work [3] Yablonskii-Vorobiev's polynomials P_n are determined by

$$P_0=P_1=1, \tag{2.9}$$

$$P_nP_{n+2}=\xi P_{n+1}^2+4P_{n+1}'^2-4P_{n+1}P_{n+1}'', \qquad n=0,1,2,\cdots. \tag{2.10}$$

These polynomials P_n have the form of (1.6) and we have expressions (1.7) and (1.9). $\{q_n, p_n\}$ satisfies the Toda equation (1.10), q_n satisfies the Painlevé-II equation (1.8) and p_n satisfies

$$p_np_n''-p_n'^2/2+4p_n^3+\xi p_n^2+(2n+1)^2/32=0. \tag{2.11}$$

Differentiating (1.8) and (2.11) we have

$$q_n'''-6q_n^2q_n'-\xi q_n'-q_n=0, \tag{2.12}$$

$$p_n'''+12p_np_n'+2\xi p_n'+p_n=0. \tag{2.13}$$

From (2.13) it follows

(2.14) $$\tilde{p}_n''' - 12\tilde{p}_n\tilde{p}_n' - \xi\tilde{p}_n' - 2\tilde{p}_n = 0.$$

(2.12) and (2.14) are the ordinary differential equations satisfied by similarity solutions of m-KdV and KdV equations. So

(2.15) $$u_n(x, t) = (3t)^{-2/3}\tilde{p}_n((3t)^{-1/3}x),$$

(2.16) $$v_n(x, t) = (3t)^{-1/3}q_n((3t)^{-1/3}x)$$

satisfy KdV and m-KdV equations respectively.

It is easy to see that u_n and v_n have expressions of (1.3) and (1.4).

§ 3. Estimate for the coefficients of F_n. The Laurent expansion at ∞

(3.1) $$\tilde{p}_{n-1} = -(\log P_n)'' = \sum_{j=0}^{\infty} (-1)^j p_{n-1,j}\xi^{-(3j+2)}$$

converges in $|\xi| > A_n$. Integrating both side of (3.1) we have

(3.2) $$P_n'/P_n = \sum_{j=0}^{\infty} (-1)^j (3j+1)^{-1} p_{n-1,j}\xi^{-(3j+1)}.$$

Inserting (1.6) into (3.2) we have

(3.3) $$P_{n,j} = (3j)^{-1}\sum_{k=1}^{j} P_{n,j-k}(-1)^{k-1}(3k+1)^{-1}p_{n-1,k}, \qquad j = 1, 2, 3, \cdots.$$

On the other hand we can show that

(3.4) $$|p_{n-1,j}| \leq ((\sqrt{5}-2)/4)(2(11+5\sqrt{5})n^2)^{j+1}, \qquad j = 0, 1, 2, \cdots.$$

Combining (3.3) and (3.4) we can show the following

Theorem 3.1 (Main theorem). *The coefficients $P_{n,j}$ of the Yablonskii-Vorobiev's polynomials P_n satisfy inequalities* (1.11).

References

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