103. On Poles of the Rational Solution of the Toda Equation of Painlevé-II Type

By Yoshinori KAMETAKA
Department of Applied Mathematics, Faculty of Engineering, Ehime University

(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 12, 1983)

§1. **Summary.** The Toda equation

(1.1) $$q_n'=p_{n-1}-p_n, \quad p_n'=p_n(q_n-q_{n+1}), \quad n=0, \pm 1, \pm 2, \cdots$$

admits the special rational solution

(1.2) $$q_n=(\log P_n/P_{n+1})', \qquad p_n=(\log P_{n+1})''-t/4$$

where

(1.3) $$P_n=\prod_{k=1}^{d(n)}(t-a_{n,k})=\sum_{j=0}^{f(n)}P_{n,j}t^{d(n)-3j}$$

are the polynomials of degree $d(n)=n(n-1)/2$ with integral coefficients ($P_{n,0}=1$, $P_{n,f(n)}\neq 0$, $f(n)=[n(n-1)/6]$). These polynomials were introduced by A. I. Yablonskii [1] and A. P. Vorobiev [2] who showed that q_n satisfies the Painlevé-II equation

(1.4) $$q_n''=2q_n^3+tq_n+n.$$

All zeros of P_n are simple, P_n and P_{n+1} have no common zero. So q_n has n^2 simple poles and p_n has $n(n+1)/2$ double poles.

A sharp estimate for the maximal modulus of these poles is obtained. $A_n=\max\{|a_{n,k}|; 1\leq k\leq d(n)\}$ satisfies

(1.5) $$n^{2/3}\leq A_{n+2}\leq 4n^{2/3} \qquad n=0, 1, 2, \cdots.$$

§2. **Recurrence relation.** If we define the rational functions q_n and p_n by the recurrence relation

(2.1) $$q_0=0, \qquad p_0=-t/4,$$

(2.2) $$q_n=(2n-1)/4p_{n-1}-q_{n-1}, \qquad p_n=-(p_{n-1}+q_n^2+t/2),$$

(2.3) $$q_{-n}=-q_n, \quad p_{-n}=p_{n-1}, \quad n=1, 2, 3, \cdots$$

then

Theorem 2.1. *$\{q_n, p_n\}$ satisfies the Toda equation* (1.1), *q_n satisfies the Painlevé*-II *equation* (1.4) *and p_n satisfies*

(2.4) $$p_np_n''-p_n'^2/2+4p_n^3+tp_n^2+(2n+1)^2/32=0$$

for every integral n.

§3. **Yablonskii-Vorobiev's polynomials.** The rational functions P_n are determined uniquely by the relation

(3.1) $$p_n=-P_nP_{n+2}/4P_{n+1}^2, \qquad n=0, \pm 1, \pm 2, \cdots$$

with initial condition

(3.2) $$P_0=P_1=1.$$

Integrating the Toda equation with respect to n we have (1.2). So we have

Theorem 3.1 (A. P. Vorobiev [2]).

(3.3) $$P_n P_{n+2} = tP_{n+1}^2 + 4P_{n+1}'^2 - 4P_{n+1}P_{n+1}''.$$

Using (2.4) and (3.3) we can show

Theorem 3.2. *P_n are the polynomials with properties stated in* §1.

§4. Laurent expansion at ∞. The Laurent expansions at ∞ for q_n and p_n are convergent in $|t|>\max\{A_n, A_{n+1}\}$ and in $|t|>A_{n+1}$ respectivery. Inserting the expressions

(4.1) $$\tilde{q}_n = q_n + nt^{-1} = \sum_{j=0}^{\infty} (-1)^j q_{n,j} t^{-(3j+4)},$$

(4.2) $$\tilde{p}_n = -p_n - t/4 = \sum_{j=0}^{\infty} (-1)^j p_{n,j} t^{-(3j+2)}$$

into the Toda equation (1.1) we have a recurrence relation for the coefficients which gives

Theorem 4.1.

(4.3) $$q_{n+1,j} \geq q_{n,j} \geq 0, \qquad p_{n,j} \geq p_{n-1,j} \geq 0,$$
$$n=0, 1, 2, \cdots, \qquad j=0, 1, 2, \cdots.$$

From these inequalities it follows

(4.4) $$A_{n+1} > A_n > A_2 = 0, \qquad n=3, 4, 5, \cdots.$$

§5. Estimate from below for A_n. $\tilde{p}_{n-1}$ has a different expression as Laurent series at ∞.

(5.1) $$\tilde{p}_{n-1} = \sum_{k=1}^{d(n)} (t - a_{n,k})^{-2} = \sum_{j=0}^{\infty} (j+1) \sum_{k=1}^{d(n)} a_{n,k}^j t^{-(j+2)}.$$

Comparing this with (4.2) we have

(5.2) $$p_{n-1,j}/(3j+1) = \sum_{k=1}^{d(n)} (-a_{n,k})^{3j}$$

where the righthand side does not exceed $d(n)A_n^{3j}$ so we have

Theorem 5.1.

(5.3) $$A_n \geq (p_{n-1,j}/(3j+1)d(n))^{1/3j}, \quad n=3, 4, 5, \cdots, \quad j=1, 2, 3, \cdots.$$

Especially

(5.4) $$A_n \geq ((n-2)(n+1))^{1/3}, \qquad n=3, 4, 5, \cdots.$$

Since we know that

(5.5) $$p_{n-1,1} = 2n(n-1)(n(n-1)-2).$$

§6. Estimate from above for A_n. From the results of §4

(6.1) $$\hat{q}_n = -n^{-1} t q_n - 1, \qquad \hat{p}_n = 1 + 4t^{-1} p_n$$

can be expressed as a convergent power series of $(-t)^{-1}$ with non negative coefficients in $|t|>A_{n+1}$. Rewriting the recurrence relation (2.1) and (2.2) we have

(6.2) $$\hat{p}_0 = 0, \qquad (\hat{q}_0 = 0),$$

(6.3) $$n\hat{q}_n = (2n-1)\hat{p}_{n-1}/(1-\hat{p}_{n-1}) - (n-1)\hat{q}_{n-1},$$
$$\hat{p}_n = 4n^2(-t)^{-3}(1+\hat{q}_n)^2 - \hat{p}_{n-1}, \qquad n=1, 2, 3, \cdots.$$

Estimating these recurrence relations inductively we have

Theorem 6.1. *Define*

$$T_n(\theta)=(4n^2\varphi(\theta))^{1/3} \quad \textit{where } \varphi(\theta)=(1+\theta)^2\theta^{-1}(1-\theta)^{-2} \tag{6.4}$$

for any fixed θ $(0<\theta<1)$. *Then we have*

$$A_{n+2}\leq T_n(\theta), \tag{6.5}$$

$$|\hat{q}_n(t)|\leq 2\theta/(1-\theta), \quad |\hat{p}_n(t)|\leq\theta \quad \textit{for } |t|\geq T_n(\theta) \tag{6.6}$$

for any $n\geq 1$.

Since $\min_{0<\theta<1}\varphi(\theta)=\varphi(\sqrt{5}-2)=(11+5\sqrt{5})/2$ then the best result is obtained when we choose $\theta=\sqrt{5}-2$. Now we arrive at our main theorem.

Theorem 6.2 (Main theorem).

$$(n(n+3))^{1/3}\leq A_{n+2}\leq(2(11+5\sqrt{5}))^{1/3}n^{2/3}, \qquad n=1,2,3,\cdots. \tag{6.7}$$

References

[1] A. I. Yablonskii: Весцi АН БССР, серыя фiз.-тэхн. Навук., no. 3 (1959).
[2] A. P. Vorobiev: On rational solutions of the second Painlevé equation. Differencial'nye Uravnenija, 1, no. 1, 79–81 (1965) (in Russian).