93. Convergence of Nonlinear Evolution Operators in Banach Spaces

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1. **Introduction.** In the recent works of Crandall and Pazy [2], Evans [3], Kobayashi *et al.* [4], and Pavel [7] has been studied the existence of an evolution operator associated with the time-dependent evolution equation

$$du(t)/dt \in A(t)u(t), \quad s<t<T, \quad u(s)=x, \tag{1}$$

where $T>0$, $s\in[0,T)$, $x\in\overline{D(A(s))}$, and $\{A(t);0\leqq t\leqq T\}$ is a family of (possibly multi-valued) nonlinear operators in a Banach space. The purpose of this note is to discuss the convergence of nonlinear evolution operators under more general conditions than those treated in [3], [4] and [7]. Our result gives an extension to the time-dependent case (1) of the convergence results for nonlinear semigroups due to Brezis and Pazy [1], Miyadera and Kobayashi [6] and Watanabe [8].

2. **Theorem.** Let X be a Banach space with norm $|\cdot|$. Let $\mathcal{A}=\{A(t);0\leqq t\leqq T\}$ be a family of nonlinear operators in X. We say that $\mathcal{A}$ is of class $G(\omega,\rho,g)$ if $\mathcal{A}$ satisfies the three conditions listed below:

(I) There exist $\omega\in(-\infty,\infty)$, a nondecreasing right-continuous function $\rho:[0,T]\to[0,\infty)$ with $\rho(0)=0$, and $g\in L^1(0,T;X)$ such that

$$\begin{aligned}(\lambda+\mu-\lambda\mu\omega)|x-u|\leqq{}&\mu|x-u-\lambda y|+\lambda|x-u+\mu v|\\&+\lambda\mu(\rho(|t-s|)+|g(t)-g(s)|)\end{aligned}\tag{2}$$

for any $\lambda>0$, $\mu>0$, $t,s\in[0,T]$, $[x,y]\in A(t)$, and $[u,v]\in A(s)$.

(II) If $t_n\in[0,T]$, $x_n\in D(A(t_n))$, $t_n\uparrow t$ and $x_n\to x$, then $x\in\overline{D(A(t))}$.

(III) For each $s\in[0,T)$ and $x\in\overline{D(A(s))}$, there exist sequences $\{t_k^n\}$, $\{x_k^n\}$ and $\{\varepsilon_k^n\}$ such that $s=t_0^n<t_1^n<\cdots<t_{N(n)}^n\leqq T$, $x_k^n\in D(A(t_k^n))$,

$$\frac{x_k^n-x_{k-1}^n}{t_k^n-t_{k-1}^n}\in A(t_k^n)x_k^n+\varepsilon_k^n,\qquad 1\leqq k\leqq N(n),$$

$$\lim\max_k(t_k^n-t_{k-1}^n)=0,\qquad \lim\sum_{k=1}^{N(n)}(t_k^n-t_{k-1}^n)|\varepsilon_k^n|=0,$$

$$\lim x_0^n=x,\qquad \lim t_{N(n)}^n=T,$$

$$\lim\sum_{k=1}^{N(n)}\int_{t_{k-1}^n}^{t_k^n}|g(\xi)-g(t_k^n)|\,d\xi=0.$$

If $\mathcal{A}$ is of class $G(\omega,\rho,g)$ then it is verified by applying the argument of [4] that there exists an evolution operator $\mathcal{U}=\{U(t,s);$

$0\leqq s\leqq t\leqq T\}$ such that $U(t,s)$ maps $\overline{D(A(s))}$ into $\overline{D(A(t))}$ for $0\leqq s\leqq t\leqq T$ and

$$(3)\qquad |U(t,s)x-z|-|x-z| \leqq\int_s^t\{[U(\xi,s)x-z,w]_+ +\omega|U(\xi,s)x-z| +\rho(|\xi-r|)+|g(\xi)-g(r)|\}d\xi$$

for every $s\in[0,\mathrm{T})$, $t\in[0,T]$, $r\in[0,T]$, $x\in\overline{D(A(s))}$ and $[z,w]\in A(r)$, where $[x,y]_+=\lim_{\lambda\downarrow 0}(|x+\lambda y|-|x|)/\lambda$ for x, $y\in X$. The evolution operator $\mathcal{U}$ is constructed through the convergence of solutions (x_k^n) of discrete schemes mentioned in condition (III), and hence for $s\in[0,T)$ and $x\in D(A(s))$ the function $u(t)\equiv U(t,s)x$ gives a weak solution of (1) in the sense of [4]. In this regard we say that $\mathcal{U}$ is an evolution operator associated with $\mathcal{A}$.

Let $\{A^m\}$ be a sequence of operators in X and define the limit operator $\mathrm{Lim}\,A^m$ of the sequence $\{A^m\}$ by the following: $[x,y]\in\mathrm{Lim}\,A^m$ if and only if there is a sequence $\{[x^m,y^m]\}$ such that $[x^m,y^m]\in A^m$ and $\lim(|x^m-x|+|y^m-y|)=0$.

Theorem. *Let $\{A(t)\}$ be of class $G(\omega,\rho,g)$ and let $\{A^m(t)\}$ of class $G(\omega^m,\rho^m,g^m)$ for $m\geqq 1$. Let $\{U(t,s)\}$ and $\{U^m(t,s)\}$ be evolution operators associated with $\{A(t)\}$ and $\{A^m(t)\}$, respectively. Suppose that $\mathrm{Lim}\,A^m(t)\supset A(t)$ for every $t\in[0,T]$, $\omega^m\leqq\omega$, $\rho^m(t)\to\rho(t)$ for every $t\in[0,T]$, $g^m\to g$ in $L^1(0,T;X)$ and $g^m(t)\to g(t)$ for every $t\in[0,T]$. Then for every $s\in[0,T)$, $x\in\overline{D(A(s))}$ and $x^m\in\overline{D(A^m(s))}$ with $x^m\to x$, we have*

$$(4)\qquad \lim U^m(t,s)x^m=U(t,s)x$$

for $s\leqq t\leqq T$ and the convergence is uniform on $[s,T]$ with respect to t.

3. Proof of Theorem. **Lemma 1.** i) *Let $0\leqq s\leqq t_0<\cdots<t_n\leqq T$ and set $\lambda_k=t_k-t_{k-1}$. Then*

$$(5)\qquad |t-t_k|-|s-t_k|\leqq\frac{1}{\lambda_k}\int_s^t(|\xi-t_{k-1}|-|\xi-t_k|)d\xi\quad(s\leqq t\leqq T,\ 1\leqq k\leqq n).$$

ii) (See [8].) *For every h, λ, and δ with $0<h\leqq\delta$, $0<h\leqq\lambda$,*

$$(6)\qquad \delta+\frac{1}{h}\int_0^t e^{\xi/h}[(\xi-\delta+h)^2+\lambda\xi]^{1/2}d\xi\leqq e^{t/h}[(t-\delta)^2+\lambda t]^{1/2}(t\geqq 0).$$

Let $\varepsilon>0$ be fixed. Then there exist $g_\varepsilon\in C([0,T];X)$ and $L_\varepsilon>0$ such that $\int_0^T|g(t)-g_\varepsilon(t)|dt<\varepsilon$ and

$$(7)\qquad \rho(|t-s|)+|g_\varepsilon(t)-g_\varepsilon(s)|\leqq L_\varepsilon|t-s|+\varepsilon,\ (t,s\in[0,T]).$$

The core of the proof of our theorem is the following.

Lemma 2. *Let $s\in[0,T)$, $x\in\overline{D(A(s))}$ and let $\{x^m\}$ be a sequence in X such that $x^m\in\overline{D(A^m(s))}$ and $x^m\to x$. Then for $x_0\in X$, $s\leqq t_0<t_1<\cdots<t_N\leqq T$, $[x_k,y_k]\in A(t_k)$ $(1\leqq k\leqq N)$, $r\in[s,T)$, $[u,v]\in A(r)$, $t\in[s,T]$ and $\lambda_k=t_k-t_{k-1}$ with $\lambda_k\in(0,1/\bar\omega)$ $(1\leqq k\leqq N)$, we have*

$$(8)\qquad \limsup|U^m(t,s)x^m-U(t,s)x| \leqq 2e^{\bar\omega(t-s)}(|x-u|+L_\varepsilon|s-r|+\varepsilon)$$

$$+2\prod_{k=1}^{N}(1-\lambda_k\bar{\omega})^{-1}[|x_0-u|+L_\varepsilon|t_0-r|+\varepsilon]$$
$$+2\prod_{k=1}^{N}(1-\lambda_k\bar{\omega})^{-1}[e^{\bar{\omega}(t-s)}((t-s-t_k+t_0)^2+\lambda(t-s))^{1/2}$$
$$\times(|v|+L_\varepsilon+|g_\varepsilon(r)-g(r)|)]$$
$$+2\prod_{k=1}^{N}(1-\lambda_k\bar{\omega})^{-1}[\sum_{k=1}^{N}(\lambda_k|g_\varepsilon(t_k)-g(t_k)|+|x_k-x_{k-1}-\lambda_k y_k|)]$$
$$+2e^{\bar{\omega}(t-s)}\int_s^t|g_\varepsilon(\xi)-g(\xi)|d\xi,$$

where $\lambda=\max_k \lambda_k$ *and* $\bar{\omega}=\max\{1,\omega\}$.

Proof. For each $i\in\{1,2,\cdots,N\}$ choose a sequence $\{[x_i^m,y_i^m]\}$ such that $[x_i^m,y_i^m]\in A^m(t_i)$ and $|x_i^m-x_i|+|y_i^m-y_i|\to 0$ as $m\to\infty$. Moreover let $\{[u^m,v^m]\}$ be any sequence such that $[u^m,v^m]\in A^m(r)$ and $|u^m-u|+|v^m-v|\to 0$ as $m\to\infty$. For simplicity in notation we use the following functions:

$p_k(t)=|U(t,s)x-x_k|+L_\varepsilon|t-t_k|+\varepsilon,\quad k=0,1,2,\cdots,$

$p_k^m(t)=|U^m(t,s)x^m-x_k^m|+L_\varepsilon|t-t_k|+\varepsilon,\quad m=1,2,\cdots,\ k=0,1,2,\cdots,$

$a_k=|x_k-x_{k-1}-\lambda_k y_k|+\lambda_k|g_\varepsilon(t_k)-g(t_k)|,\quad k=1,2,\cdots,$

$b=|v|+L_\varepsilon+|g_\varepsilon(r)-g(r)|,\quad \alpha_k=1/\lambda_k-\bar{\omega},$

and define $q_k(t)$ by

$$q_k(t)=e^{\bar{\omega}(t-s)}(|x-u|+L_\varepsilon|s-r|+\varepsilon)$$
$$+\prod_{i=1}^{k}(1-\lambda_i\bar{\omega})^{-1}[|x_0-u|+L_\varepsilon|t_0-r|+\varepsilon+\sum_{i=1}^{k}a_i]$$
$$+\prod_{i=1}^{k}(1-\lambda_i\bar{\omega})^{-1}[e^{\bar{\omega}(t-s)}((t-s-\sum_{i=1}^{k}\lambda_i)^2+\lambda(t-s))^{1/2}b]$$
$$+e^{\bar{\omega}(t-s)}\int_s^t|g_\varepsilon(\xi)-g(\xi)|d\xi$$

for $k=1,2,3,\cdots$, and $t\in[s,T]$. We shall estimate $p_k(t)$ and $p_k^m(t)$ by induction on k. For the values $p_k(t)$ we demonstrate that

$$p_k(t)\leqq q_k(t) \tag{9}$$

for $k\geqq 1$. First we have

$$p_0(t)\leqq e^{\bar{\omega}(t-s)}(|x-u|+L_\varepsilon|s-r|+\varepsilon)+(|x_0-u|+L_\varepsilon|t_0-r|+\varepsilon)$$
$$+e^{\bar{\omega}(t-s)}(t-s)b+e^{\bar{\omega}(t-s)}\int_s^t|g_\varepsilon(\xi)-g(\xi)|d\xi.$$

On the other hand, the inequalities (3), (5) and (7) together imply that

$$p_k(t)\leqq p_k(s)-\alpha_k\int_s^t p_k(\xi)d\xi+\frac{1}{\lambda_k}\int_s^t p_{k-1}(\xi)d\xi \tag{10}$$
$$+(t-s)a_k/\lambda_k+\int_s^t|g_\varepsilon(\xi)-g(\xi)|d\xi.$$

From this it follows that

$$\exp[\alpha_k(t-s)]p_k(t)\leqq p_k(s)+\frac{1}{\lambda_k}\int_s^t\exp[\alpha_k(\xi-s)]p_{k-1}(\xi)d\xi \tag{11}$$
$$+(1-\lambda_k\bar{\omega})^{-1}[\exp[\alpha_k(t-s)]-1]a_k$$
$$+\int_s^t\exp[\alpha_k(\xi-s)]|g_\varepsilon(\xi)-g(\xi)|d\xi.$$

On the other hand, condition (I) implies that

$$p_k(s)\leqq|x-u|+L_\varepsilon|s-r|+\varepsilon \tag{12}$$
$$+\prod_{i=1}^{k}(1-\lambda_i\bar{\omega})^{-1}[|x_0-u|+L_\varepsilon|t_0-r|+\varepsilon+\sum_{i=1}^{k}a_i+(t_k-t_0)b].$$

Combining (11) with (12), we have

$$(13)\quad \exp[\alpha_k(t-s)]p_k(t)$$
$$\leqq \frac{1}{\lambda_k}\int_s^t \exp[\alpha_k(\xi-s)]p_{k-1}(\xi)d\xi+|x-u|+L_\varepsilon|s-r|+\varepsilon$$
$$+\prod_{i=1}^k(1-\lambda_i\bar{\omega})^{-1}[|x_0-u|+L_\varepsilon|t_0-r|+\varepsilon+(t_k-t_0)b$$
$$+\sum_{i=1}^{k-1}a_i+\exp[\alpha_k(t-s)]a_k]$$
$$+\int_s^t \exp[\alpha_k(\xi-s)]|g_\varepsilon(\xi)-g(\xi)|d\xi.$$

Now suppose that (9) holds for $k-1$. Replacing $p_{k-1}(t)$ on the right side of (13) with $q_{k-1}(t)$ and then applying (6) with $h=\lambda_k$ and $\delta=\sum_{i=1}^k\lambda_i$, we infer that $p_k(t)$ is bounded by $q_k(t)$. The proof of (9) is thereby complete.

In a manner similar to the derivation of (9), we obtain

$$(14)\quad p_k^m(t)\leqq q_k(t)+e^{\bar{\omega}(t-s)}[|x^m-x|+|u^m-u|+T|v^m-v|]$$
$$+\prod_{i=1}^k(1-\lambda_i\bar{\omega})^{-1}\sum_{i=0}^k|x_i^m-x_i|$$
$$+\prod_{i=1}^k(1-\lambda_i\bar{\omega})^{-1}[\sum_{i=1}^k(|x_i^m-x_{i-1}^m-\lambda_i y_i^m|-|x_i-x_{i-1}-\lambda_i y_i|)]$$
$$+e^{\bar{\omega}(t-s)}\sum_{i=1}^k\int_s^t[|\rho^m(|\xi-t_i|)-\rho(|\xi-t_i|)|+|g^m(\xi)-g(\xi)|$$
$$+|g^m(t_i)-g(t_i)|]d\xi$$
$$+e^{\bar{\omega}(t-s)}\int_s^t[|\rho^m(|\xi-r|)-\rho(|\xi-r|)|+|g^m(\xi)-g(\xi)|$$
$$+|g^m(r)-g(r)|]d\xi.$$

Since

$$|U^m(t,s)x^m-U(t,s)x|\leqq|U^m(t,s)x^m-x_k^m|+|U(t,s)x-x_k|+|x_k^m-x_k|,$$

the estimates (9) and (14) together imply the desired estimate (8).

Remark. Suppose that

$R(I-\lambda A(t+\lambda))\supset\overline{D(A(t))}$ for every $t\in[0,T)$ and $\lambda>0$ with $t+\lambda\leqq T$, then the conclusion of the theorem holds without the assumption that $g^m(t)\to g(t)$ for every point $t\in[0,T]$.

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