10. On Surfaces of Class VII_0 with Global Spherical Shells

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Introduction. A minimal compact complex surface is called a surface of class VII_0 or in short a VII_0 surface if b_1 (the first Betti number) is equal to 1. Some VII_0 surfaces, i.e., Hopf surfaces and Inoue surfaces with $b_2>0$ are smooth deformations of singular rational surfaces, each with a double curve, as was observed by [4], [7]. The purpose of this note is to report some results on smooth deformations of singular rational surfaces, each with a double curve [6]. As an application of them we study VII_0 surfaces with global spherical shells in the sense of [3]. In § 4 we shall give a partial classification of surfaces with global spherical shells by using the results of [2], [5].

Notations. We denote by S a compact complex surface, by $\mathcal{O}_S$ the structure sheaf of S, by b_i the i-th Betti number of S. For a singular complex surface Y we denote also by $\mathcal{O}_Y$ the structure sheaf of Y. For two divisors C and C' we denote by CC' the intersection number of C and C', by C^2 the selfintersection number of C. We denote by $\mathcal{S}_t$ a fiber $\pi^{-1}(t)$ $(t\in D)$ of a morphism $\pi:\mathcal{S}\to D$, by $\mathcal{D}_t$ the intersection of $\mathcal{D}$ and $\mathcal{S}_t$ for a π-flat Cartier divisor $\mathcal{D}$ of $\mathcal{S}$. We denote by F_m a Hirzebruch surface $\mathrm{Proj}\,(\mathcal{O}_{P^1}\oplus\mathcal{O}_{P^1}(m))$. We mean by a chain of rational curves a reduced divisor D with a decomposition $D=\sum_{\nu=1}^{r}D_\nu$ such that D_ν is a nonsingular rational curve, $D_\nu D_{\nu+1}=1$, $D_\nu D_\mu=0$ $(\nu\neq\mu, \mu\pm1)$. Then $Zykel\,(D)=(-D_1^2,\cdots,-D_r^2)$. For a cycle $C=\sum_{\nu=1}^{r}C_\nu$ of rational curves with $C_\nu C_{\nu+1}=1$ $(\nu\in Z/rZ)$, we denote $Zykel\,(C)=(-C_1^2,\cdots,-C_r^2)$ $(r\geqq2)$; $(-C_1^2+2)$ $(r=1)$. See [5] for the definition of a cycle of rational curves.

§ 1. Global spherical shells.

(1.1) **Definition** [3]. A subset Σ of a compact complex surface S is called a global spherical shell (abbr. a GSS) if

1) Σ is a spherical shell $\{(z_1,z_2)\in C^2;\ r<|z_1^2|+|z_2^2|<R\}$ for some positive numbers r and R, and

2) the complement $S-\Sigma$ of Σ in S is connected.

(1.2) **Theorem** [3]. *Any surface with a GSS is a deformation of a blown-up primary Hopf surface with $b_1=1$. A minimal surface with a GSS is a primary Hopf surface if and only if $b_2=0$.*

(1.3) **Theorem** [3], [6]. *Any Inoue surface with $b_2>0$ contains a GSS.*

See [5] for the definition of Inoue surfaces with $b_2>0$.

§ 2. Smooth deformations of singular rational surfaces.

(2.1) **Definition.** A triple (X, C_1, C_2) or a quadruple (X, C_1, C_2, σ) is called *admissible* if

1) X is a nonsingular compact complex surface,

2) C_ν $(\nu=1,2)$ is a nonsingular rational curve on X with $C_1^2>0$, $C_1C_2=0$, $C_1^2+C_2^2\geqq 0$,

3) σ is an isomorphism of C_1 onto C_2.

Moreover we call the triple or the quadruple *minimal* if any exceptional curve of the first kind meets either C_1 or C_2.

By identifying C_1 and C_2 by σ, we have a singular surface $Y=X/\sigma$ with a double curve $\bar{C}$, a nonsingular rational curve, along which Y is locally given by $xy=0$ with suitable local coordinates.

(2.2) **Theorem [6].** *Any minimal admissible triple* (X, C_1, C_2) *with* $C_1^2=m$, $C_2^2=-n$ $(m\geqq n>0)$ *is one of the following;*

1) $m=n=1$, *X is a finite succession of blowing-ups of $\boldsymbol{P}^2$ with centers over the previous centers outside of a fixed line l, C_1 and C_2 are a proper transform of l and the exceptional curve of the last blowing-up with* $C_2^2=-1$.

2) $m\geqq 2$, *X is a finite succession of blowing-ups of a Hirzebruch surface $\boldsymbol{F}_k$ of degree k ($k\equiv m \bmod 2$, $k\leqq m$) with centers over the previous centers outside of a fixed section τ of $\boldsymbol{F}_k$ over $\boldsymbol{P}^1$ with $\tau^2=m$, C_1 and C_2 are proper transforms of τ and an exceptional curve of the first kind by one of the blowing-ups.*

3) $m=n\geqq 2$, *$X=\boldsymbol{F}_m$, C_1 and C_2 are sections of $\boldsymbol{F}_m$ over $\boldsymbol{P}^1$.*

4) $m=4$, *X is a finite succession of blowing-ups of $\boldsymbol{P}^2$ with centers over the previous centers outside of a quadric q, C_1 and C_2 are proper transforms of q and an exceptional curve of the first kind by one of the blowing-ups.*

5) $m=n=2$, *X is a three-times blowing-up $Q_{l\cap q}Q_p(\boldsymbol{P}^2)$ of $\boldsymbol{P}^2$, C_1 and C_2 are proper transforms of a quadric q and a line l where P is a point of $l-l\cap q$, l and q may have a contact.*

(2.3) **Theorem [6].** *Let (X, C_1, C_2, σ) be a minimal admissible quadruple. Then $Y=X/\sigma$ is smoothable by flat deformation. More precisely,* $\mathrm{Ext}^2_{\mathcal{O}_Y}(L^{\cdot}_Y, \mathcal{O}_Y)=0$ *where $L^{\cdot}_Y$ is the cotangent complex of Y. And there exists a proper flat family $\pi: \mathcal{S}\to D$ over the disc D such that $\mathcal{S}_0=Y$, $\mathcal{S}_t$ $(t\neq 0)$ is a smooth surface with $b_1=1$. If moreover $C_1^2=1$, $C_2^2=-1$, then $\mathcal{S}_t$ $(t\neq 0)$ is a* VII_0 *surface with a GSS for $|t|$ small.*

(2.4) **Theorem [6].** *Let S be a minimal surface with a GSS. Then there exist a minimal admissible quadruple (X, C_1, C_2, σ) with $C_1^2=1$, $C_2^2=-1$ and a proper flat family $\pi: \mathcal{S}\to D$ over the unit disc with a π-flat Cartier divisor $\mathcal{D}$ of $\mathcal{S}$ such that*

1) $\mathcal{S}_0 \simeq X/\sigma$, $\mathcal{S}_{t_0} \simeq S$ *for some* t_0 *in* D,

2) $\mathcal{S}_t$ $(t \neq 0)$ *is a minimal surface with a GSS*,

3) $\mathcal{D}_t$ *is the maximal reduced effective divisor of* $\mathcal{S}_t$ $(t \neq 0)$ *whose dual graph is independent of* t $(\neq 0)$.

§ 3. Curves on surfaces with GSS's.

(3.1) **Theorem** (Kato, see [1], [6]). *Let* S *be a surface with a GSS. Then* $b_2(S) = \sharp$ *(irreducible rational curves on* S*).*

(3.2) **Theorem.** *Any minimal surface with a GSS and* $b_2 > 0$ *has a cycle of rational curves and only finitely many irreducible curves. A reduced effective divisor* D *is the maximal reduced effective divisor on a minimal surface with a GSS and* $b_2 > 0$ *if and only if* D *is one of the following;*

1) $D = E + Z$, E *is a nonsingular elliptic curve with* $E^2 = -n$, Z *is a cycle of* n *rational curves* Z_ν *with* $Z_\nu^2 = -2$ $(n \geqq 2)$, $Z_1^2 = 0$ $(n = 1)$,

2) $D = A + B$, A *and* B *are cycles of rational curves and*

$$Zykel\,(A) = (p_1, \underbrace{2, \cdots, 2}_{(q_1-3)}, p_2, \cdots, p_n, \underbrace{2, \cdots, 2}_{(q_n-3)})$$

$$Zykel\,(B) = (\underbrace{2, \cdots, 2}_{(p_1-3)}, q_1, \cdots, q_{n-1}, \underbrace{2, \cdots, 2}_{(p_n-3)}, q_n)$$

for certain positive integers p_j, q_j $(\geqq 3)$, n $(\geqq 1)$.

3) D *is a cycle of* n *rational curves* C_ν *with* $C_\nu^2 = -2$, $(n \geqq 2)$, $C_1^2 = 0$ $(n = 1)$,

4) D *is a cycle* C *of rational curve with* $C^2 < 0$ *and*

$$Zykel\,(C) = (p_1, \underbrace{2, \cdots, 2}_{(q_1-3)}, p_2, \cdots, p_n, \underbrace{2, \cdots, 2}_{(q_n-3)}, p_{n+1}$$

$$\underbrace{2, \cdots, 2}_{(p_1-3)}, q_1, \underbrace{2, \cdots, 2}_{(p_2-3)}, q_2, 2, \cdots, 2, q_n, \underbrace{2, \cdots, 2}_{(p_{n+1}-3)})$$

for certain positive integers p_j, q_j $(\geqq 3)$, n $(\geqq 1)$,

5) *there is a decomposition* $D = \sum_{k=1}^m (C_k + D_k)$ *such that*

i) $C' = \sum_{k=1}^m C_k$ *is a cycle of rational curves with* $(C')^2 < 0$, C_k $(m \geqq 2)$ *and* D_k *are nonempty chains of nonsingular rational curves,* C_1 *is a cycle of rational curves* $(m = 1)$

ii) C_k *and* D_k *are dual in the sense that*

$$Zykel\,(C_k) = (p_1, \underbrace{2, \cdots, 2}_{(q_1-3)}, p_2, \underbrace{2, \cdots, 2}_{(q_2-3)}, \cdots, p_{n-1}, \underbrace{2, \cdots, 2}_{(q_{n-1}-3)}, p_n)$$

$$Zykel\,(D_k) = (\underbrace{2, \cdots, 2}_{(p_n-2)}, q_{n-1}, \underbrace{2, \cdots, 2}_{(p_{n-1}-3)}, q_{n-2}, \cdots, q_1, \underbrace{2, \cdots, 2}_{(p_1-3)})$$

for certain positive integers n $(\geqq 1)$, p_n $(\geqq 2)$, p_j, q_j $(\geqq 3, 1 \leqq j \leqq n-1)$ *or*

$$Zykel\,(C_k) = (p_1, \underbrace{2, \cdots, 2}_{(q_1-3)}, p_2, \underbrace{2, \cdots, 2}_{(q_2-3)}, \cdots, p_{n-1}, \underbrace{2, \cdots, 2}_{(q_{n-1}-3)}, p_n)$$

$$Zykel\,(D_k) = (\underbrace{2, \cdots, 2}_{(p_n-2)}, q_{n-1}, \underbrace{2, \cdots, 2}_{(p_{n-1}-3)}, q_{n-2}, \cdots, q_2, \underbrace{2, \cdots, 2}_{(p_2-3)})$$

for certain positive integers n $(\geqq 2)$, $p_1 = 2$, p_j, q_j, q_1 $(\geqq 3, 2 \leqq j \leqq n-1)$ p_n $(\geqq 2)$ *where* p_j *and* q_j *depend on* k, $p_n \geqq 3$ *if* $p_1 = 2$, $n = 2$

iii) $C_kD_{j-1}=\delta_{jk}$, $C_kC_{k+1}=1$ ($j, k\in Z/mZ$), *and the last irreducible component of* C_k *and the first of* D_k *meet* C_{k+1} *at distinct points of the first irreducible component of* C_{k+1} *transversally, if* $m\geqq 2$. *If* $m=1$, *then* $C_1D_1=1$ *and the first irreducible component of* C_1 *meet the first of* D_1.

(3.3) Following [1] we define an invariant $\sigma(S)$ by

$\sigma(S)=-\sum_{E:\text{ irred curve on } S} E^2+2\sharp$ {rational curves with nodes}.

§4. A partial classification and related results.

(4.1) **Theorem** [2]+[6]. *Let S be a minimal surface with a GSS and* $b_2>0$. *Then the following are true.*

1) *If S has an elliptic curve and a cycle of rational curves, then S is a parabolic Inoue surface.*

2) *If S has two cycles of rational curves, then S is a hyperbolic Inoue surface.*

3) *If S has a cycle C of n rational curves with* $C^2=0$, *then S is an exceptional compactification of an affine bundle of degree n.*

4) *If S has a cycle C of rational curves with* $C^2<0$ *and there is no curve except irreducible components of C, then S is a half Inoue surface.*

See [2], [5] for the definition of exceptional compactifications.

(4.2) **Theorem.** *Let S be a minimal surface with a GSS and* $b_2>0$. *Then* $3b_2\geqq\sigma(S)\geqq 2b_2$. (See [1].) *Moreover*

1) $\sigma(S)=3b_2$ *if and only if S is either a parabolic or a hyperbolic or a half Inoue surface, and*

2) $\sigma(S)=2b_2$ *if and only if S is an exceptional compactification of an affine bundle of degree* b_2.

(4.3) **Theorem.** *Let S be a* VII_0 *surface with* $b_2>0$, *D the maximal reduced divisor of S. Suppose that D is connected. Then* $\sigma(S)\leqq 3b_2$. *Equality holds if and only if S is a half Inoue surface.*

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