79. On q-Additive Functions. I

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(Communicated by Shokichi IYANAGA, M. J. A., June 14, 1983)

1. Let q be an arbitrary fixed natural number $\geqq 2$. Then a natural number n can be written in the unique way:

$$n=\sum_{k=0}^{\infty} a_k(n)q^k, \quad 0\leqslant a_k(n)\leqslant q-1 \quad (q\text{-adic expansion of } n).$$

We say that an arithmetic function $g(n)$ is *q-additive*, if

$$(1) \qquad g(0)=0 \quad \text{and} \quad g(n)=\sum_{k=0}^{\infty} g(a_k(n)q^k)$$

whenever $n=\sum_{k=0}^{\infty} a_k(n)q^k$ (cf. Gelfond [1]).***) The function "Sum of digits" $S_q(n)$ defined by $S_q(n)=\sum_{k=0}^{\infty} a_k(n)$, is a typical example of a q-additive function.

Let $[x]$ denote the integral part of x, and $\zeta(s, r/q)$, $1\leqslant r\leqslant q$ the Hurwitz zeta function defined by $\zeta(s, r/q)=\sum_{m=0}^{\infty}(m+r/q)^{-s}$ for Re (s) >1. We put

$$\mathcal{A}=\Big\{g(n): q\text{-additive function such that the convergence abscissa of } \int_1^{\infty} g([t])t^{-s-1}dt<\infty\Big\},$$

$$\mathcal{B}=\{H(z): \text{Taylor series in } z \text{ with positive radius of convergence}\}.$$

In this article we give a result concerning a relation between $\mathcal{A}$ and $\mathcal{B}$. Our theorem is:

Theorem. *For q given functions $H_r(z)\in\mathcal{B}$, $1\leqslant r\leqslant q$, there exist a unique $g(n)\in\mathcal{A}$ and a unique $H(z)\in\mathcal{B}$ such that*

$$(2) \qquad \sum_{r=1}^{q} H_r(q^{-s})\zeta\Big(s, \frac{r}{q}\Big)=s\cdot q^s\cdot\int_1^{\infty} g([t])t^{-s-1}dt+q^{s-1}H(q^{-s})\zeta(s).$$

Conversely, for a given $g(n)\in\mathcal{A}$ and an $H(z)\in\mathcal{B}$, there exists a unique system $H_r(z)\in\mathcal{B}$, $1\leqslant r\leqslant q$, which satisfies (2).

We intend to give, as an application of this result, an explicit summation formula $\sum_{n\leqslant x} g(n)$ for some q-additive functions, in a subsequent article.

2. The following lemma plays an important part in the proof of our Theorem.

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***) The values of g on the set $\{rq^k: 1\leqslant r\leqslant q-1,\ k\in N\}$, determine completely the q-additive function $g(n)$.

Lemma (Functional equation involving a q-additive function). *If $g(n) \in \mathcal{A}$, then*

$$s \cdot q^s \cdot \int_1^\infty g([t]) t^{-s-1} dt = \sum_{r=1}^{q-1} \left(\zeta\left(s, \frac{r}{q}\right) - \zeta\left(s, \frac{r+1}{q}\right) \right) f_r(q^{-s}),$$

where $f_r(z) = \sum_{k=0}^\infty g(rq^k) z^k \in \mathcal{B}$, $1 \leqslant r \leqslant q-1$.

We sketch the proof of Lemma. Here we only consider a q-additive function $g(n)$ satisfying $g(n) = O(n)$. Then the series in two variables, $\sum_{n=1}^\infty b^{g(n)} x^n$, converges in some neighbourhood of $(b, x) = (1, 0)$, and is equal to

$$\prod_{k=0}^\infty \left(1 + \sum_{r=1}^{q-1} b^{g(rq^k)} x^{rq^k}\right).$$

The equation

$$\left\{ \frac{\partial}{\partial b} \sum_{n=1}^\infty b^{g(n)} x^n \right\}_{b=1} = \left\{ \frac{\partial}{\partial b} \left(\prod_{k=0}^\infty \left(1 + \sum_{r=1}^{q-1} b^{g(rq^k)} x^{rq^k}\right)\right)\right\}_{b=1},$$

gives

$$\sum_{n=1}^\infty (g(n) - g(n-1)) x^n = \sum_{k=0}^\infty \left\{ \frac{1 - x^{q^k}}{1 - x^{q^{k+1}}} \left(\sum_{r=1}^{q-1} g(rq^k) x^{rq^k} \right) \right\}, \qquad |x| \leqslant 1.$$

We make here Mellin transform of both sides and have

$$s \cdot \int_1^\infty g([t]) t^{-s-1} dt = \sum_{r=1}^{q-1} \left\{ \zeta\left(s, \frac{r}{q}\right) - \zeta\left(s, \frac{r+1}{q}\right) \right\} \left(\sum_{k=0}^\infty g(rq^k) q^{-s(k+1)} \right).$$

Since $g(n) \in \mathcal{A}$, it is easily seen that the function $f_r(z) = \sum_{k=0}^\infty g(rq^k) z^k$ belongs to $\mathcal{B}$, and this proves our functional equation. For a q-additive function $g(n)$ such that $g(n) \neq O(n)$, we can prove our formula through making use of a q-additive function $\tilde{g}_\alpha(n)$ defined by

$$\tilde{g}_\alpha(rq^k) = |g(rq^k)| q^{-\alpha k},$$

where

$$\alpha > \max_{1 \leqslant r \leqslant q-1} \left\{ \text{Real part of the absolute convergence abscissa of } \sum_{k=0}^\infty g(rq^k) q^{-sk} \right\}.$$

3. Now we sketch the proof of our Theorem.

1°. It is sufficient to prove that, under the condition $\sum_{r=1}^q H_r(z) = 0$, there exists a unique $g(n) \in \mathcal{A}$ such that

$$\sum_{r=1}^q H_r(q^{-s}) \zeta\left(s, \frac{r}{q}\right) = s \cdot q^s \cdot \int_1^\infty g([t]) t^{-s-1} dt.$$

In fact, for a given $\{H_r(z)\}_{r=1}^q$, we put

$$F(z) = \sum_{r=1}^q H_r(z) \quad \text{and} \quad \tilde{H}_r(z) = H_r(z) - \frac{1}{q} F(z),$$

then $\{\tilde{H}_r(z)\}_{r=1}^q$ satisfy

$$\sum_{r=1}^q \tilde{H}_r(z) = 0, \quad \text{and} \quad \sum_{r=1}^q \left\{ \frac{1}{q} F(z) \zeta\left(s, \frac{r}{q}\right) \right\} = q^{s-1} \zeta(s) F(z).$$

2°. Since $\sum_{r=1}^q H_r(z) = 0$, we can transform the left-hand side of (2) into

$$\sum_{r=1}^{q} H_r(q^{-s})\zeta\left(s,\ \frac{r}{q}\right)=\sum_{r=1}^{q-1}\left(\zeta\left(s,\ \frac{r}{q}\right)-\zeta\left(s,\ \frac{r+1}{q}\right)\right)A_r(q^{-s})$$

where $A_r(z)=\sum_{i\leqslant r} H_i(z)\in\mathcal{B}$.

3°. For a given system $B_r(z)\in\mathcal{B}$, $1\leqslant r\leqslant q-1$, we write

$$q^{-s}B_r(q^{-s})=\sum_{k=0}^{\infty} b_r(k)q^{-s(k+1)},\qquad 1\leqslant r\leqslant q-1,$$

and put $f(rq^k)=b_r(k)$, for $1\leqslant r\leqslant q-1$, $k\geqslant 0$, and these determine the q-additive function $f(n)$. From the condition $B_r(z)\in\mathcal{B}$ for any r, we can prove that $f(n)\in\mathcal{A}$. Then our Lemma shows that

$$(3)\qquad s\cdot q^s\cdot\int_1^\infty f([t])t^{-s-1}dt=\sum_{r=1}^{q-1}\left(\zeta\left(s,\ \frac{r}{q}\right)-\zeta\left(s,\ \frac{r+1}{q}\right)\right)B_r(q^{-s}),$$

and the uniqueness of such $f(n)$ can be proved easily. Conversely, for a q-additive function $f(n)\in\mathcal{A}$, we can define the $q-1$ functions $B_r(z)\in\mathcal{B}$, $1\leqslant r\leqslant q-1$, by $B_r(z)=\sum_{k=0}^\infty f(rq^k)z^k$, which satisfy (3). So, by this one-to-one correspondence, we can find a unique q-additive function $g(n)\in\mathcal{A}$, for the system $\{A_r(z)\}_{r=1}^{q-1}$ which appeared in 2°, with the property

$$s\cdot q^s\cdot\int_1^\infty g([t])t^{-s-1}dt=\sum_{r=1}^{q} H_r(q^{-s})\zeta\left(s,\ \frac{r}{q}\right).$$

4°. *Proof of the converse.* We start from a given $g(n)\in\mathcal{A}$ and a given $H(z)\in\mathcal{B}$, and put

$$\begin{cases} H_1(z)=f_1(z)-\dfrac{1}{q}H(z), \\ H_r(z)=f_r(z)-f_{r-1}(z)-\dfrac{1}{q}H(z), \qquad 2\leqslant r\leqslant q-1, \\ H_q(z)=-f_{q-1}(z)-\dfrac{1}{q}H(z), \end{cases}$$

where $f_r(z)=\sum_{k=0}^\infty g(rq^k)z^k$, $1\leqslant r\leqslant q-1$. Then these q functions $H_r(z)$, $1\leqslant r\leqslant q$, satisfy (2) and every $H_r(z)$ is in $\mathcal{B}$. The uniqueness of the system $\{H_r(z)\}_{r=1}^q$ is verified easily and this proves our Theorem.

Reference

[1] A. O. Gelfond: Sur les nombres qui ont des propriétés additives et multiplicatives données. Acta Arithmetica, **13**, 259–265 (1967/68).