26. Remarks on the Uniqueness in an Inverse Problem for the Heat Equation. I

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§1. **Introduction.** For $(p, h, H, a) \in C^1[0,1] \times \mathcal{R} \times \mathcal{R} \times L^2(0,1)$, let $(E_{p,h,H,a})$ denote the heat equation

(1.1) $$\frac{\partial u}{\partial t} + \left(p(x) - \frac{\partial^2}{\partial x^2}\right)u = 0 \qquad (0<t<\infty, 0<x<1)$$

with the boundary condition

(1.2) $$\frac{\partial u}{\partial x} - hu_{|x=0} = 0, \quad \frac{\partial u}{\partial x} + Hu_{|x=1} = 0 \quad (0<t<\infty)$$

and with the initial condition

(1.3) $$u_{|t=0} = a(x) \qquad (0<x<1).$$

Let $A_{p,h,H}$ be the realization in $L^2(0,1)$ of the differential operator $p(x) - (\partial^2/\partial x^2)$ with the boundary condition (1.2), and let $\{\lambda_n \mid n=0,1,\cdots\}$ and $\{\phi(\cdot, \lambda_n) \mid n=0,1,\cdots\}$ be the eigenvalues and the eigenfunctions of $A_{p,h,H}$, respectively, the latter being normalized by $\phi(0,\lambda_n)=1$ ($n=0, 1, \cdots$). Noting that each λ_n ($n=0,1,\cdots$) is simple, we call

$$N = \sharp\{\lambda_n \mid (a, \phi(\cdot, \lambda_n)) = 0\}$$

the "degenerate number" of $a \in L^2(0,1)$ with respect to $A_{p,h,H}$, where $(\ ,\ )$ means the L^2-inner product.

Let T_1, T_2 in $0 \leqq T_1 < T_2 < \infty$ be given. For the solution $u=u(t,x)$ of the equation $(E_{p,h,H,a})$, the following theorem was proved by Murayama [1] and Suzuki [4], differently:

Theorem 0. *The equality*

(1.4′) $$v(t,\xi) = u(t,\xi) \qquad (T_1 \leqq t \leqq T_2\,;\ \xi = 0, 1)$$

implies

(1.5) $$(q, j, J, b) = (p, h, H, a)$$

if and only if $N=0$, *where* $v=v(t,x)$ *is the solution of* $(E_{q,j,J,b})$ $((q,j,J,b) \in C^1[0,1] \times \mathcal{R} \times \mathcal{R} \times L^2(0,1))$.

In the present paper, for $x_0 \in (0,1]$, we consider

(1.4) $$v_x(t,x_0) = u_x(t,x_0), \quad v(t,\xi) = u(t,\xi) \quad (T_1 \leqq t \leqq T_2\,;\ \xi = 0, x_0)$$

instead of (1.4′), and study

Problem. *Under what condition on* (p, h, H, a), *does* (1.4) *imply* (1.5)?

Namely, we show when

(1.6) $$\hat{\mathcal{M}} = \{(p, h, H, a)\}$$

is satisfied, where $\hat{\mathcal{M}} = \{(q, j, J, b) \mid C^1[0,1] \times \mathcal{R} \times \mathcal{R} \times L^2(0,1) \mid$ (1.4) holds for the solution $v=v(t,x)$ of the equation $(E_{q,j,J,b})\}$. In this problem, the position of x_0 plays an important role:

Theorem 1. *In the case of $x_0=1$, (1.6) holds if and only if $N=0$.*

Theorem 2. *In the case of $1/2<x_0<1$, (1.6) holds if $N<\infty$.*

Theorem 3. *In the case of $x_0=1/2$, (1.6) holds if and only if $N\leqq 1$.*

Theorem 4. *In the case of $0<x_0<1/2$, we always have $\hat{\mathcal{M}} \supsetneqq \{(p,h,H,a)\}$.*

If $x_0=1$, (1.4) is equivalent to (1.4′) and $J=H$, unless $a\equiv 0$. Hence Theorem 1 follows from Suzuki [4, Theorem 1]. In the present paper we prove Theorems 2–4. The proof suggests the following facts, though details are omitted: (I) *$q(x)=p(x)$ $(0\leqq x\leqq x_0)$ and $j=h$ follow from $N<\infty$ and (1.4), whenever $0<x_0<1$.* (II) *If $x_0\neq 1$, in any case (1.5) doesn't hold without $v_x(t,x_0)=u_x(t,x_0)$ in (1.4).*

§ 2. Preliminaries. Let $\Omega\subset\mathcal{R}^2$ be the interior of a triangle $\triangle$OAB with $\overline{\mathrm{OA}}=\overline{\mathrm{OB}}$, $\angle\mathrm{AOB}=\pi/2$, AB being parallel to either the x-axis or the y-axis and let $r\in C^1(\bar{\Omega})$ be given. We get the following propositions on the hyperbolic equation

$$K_{xx}-K_{yy}=rK \qquad (\text{on } \bar{D}) \tag{2.1}$$

in the same way as in Picard [2], where ν means the outer unit normal vector on $\partial\Omega$:

Proposition 1. *For given $f\in C^2(\overline{\mathrm{OA}})$ and $g\in C^2(\overline{\mathrm{OB}})$ with $f_{|0}=g_{|0}$, there exists a unique $K\in C^2(\bar{\Omega})$ such that (2.1) and*

$$K_{|\mathrm{OA}}=f, \qquad K_{|\mathrm{OB}}=g. \tag{2.2}$$

Proposition 2. *For given $f\in C^2(\overline{\mathrm{AB}})$ and $g\in C^1(\overline{\mathrm{AB}})$, there exists a unique $K\in C^2(\bar{\Omega})$ such that (2.1) and*

$$K_{|\mathrm{AB}}=f, \qquad \frac{\partial}{\partial\nu}K_{|\mathrm{AB}}=g. \tag{2.2′}$$

Proposition 3. *For given $f\in C^2(\overline{\mathrm{OA}})$, $g\in C^1(\overline{\mathrm{AB}})$ and $h\in\mathcal{R}$, there exists a unique $K\in C^2(\bar{\Omega})$ such that (2.1) and*

$$K_{|\mathrm{OA}}=f, \qquad \frac{\partial}{\partial\nu}K+hK_{|\mathrm{AB}}=g. \tag{2.2″}$$

These equations are reduced to certain integral equations of Volterra type, and are solved by the iteration.

Let $\phi=\phi(x)=\phi(x,\lambda)$ $(\lambda\in\mathcal{R})$ be the solution of

$$\left(\frac{d^2}{dx^2}+\lambda\right)\phi=p(x)\phi, \quad \phi(0,\lambda)=1, \quad \phi'(0,\lambda)=h. \tag{2.3}$$

This notation is compatible to that of $\phi(\cdot,\lambda_n)$ in § 1. Put $D=\{(x,y)\,|\,0<y<x<1\}$. The following Lemma 1 is shown by Propositions 1 and 3, while Lemma 2 is obtained in the same way as in Suzuki-Murayama [3]. See Suzuki [4], [5], for details.

Lemma 1. *For given p, $q\in C^1[0,1]$ and h, $j\in\mathcal{R}$, there exists a* *unique* $K \in C^2(\bar{D})$ *such that*

(2.4.a) $$K_{xx}-K_{yy}+p(y)K=q(x)K \qquad (\text{on } \bar{D})$$

(2.4.b) $$K(x,x)=(j-h)+\frac{1}{2}\int_0^x (q(s)-p(s))ds \qquad (0\leqq x\leqq 1)$$

(2.4.c) $$K_y(x,0)=hK(x,0) \qquad (0\leqq x\leqq 1).$$

Lemma 2. *For K in Lemma* 1,

(2.5) $$\psi(x,\lambda)=\phi(x,\lambda)+\int_0^x K(x,y)\phi(y,\lambda)dy$$

satisfies

(2.6) $$\left(\frac{d^2}{dx^2}+\lambda\right)\psi=q(x)\psi, \quad \psi(0,\lambda)=1, \quad \psi'(0,\lambda)=j.$$

Let $\{n_l\}_{l=1}^N$ $(n_1<n_2<\cdots<n_N)$ be a finite set of non-negative integers. We then have

Lemma 3. $\{\phi(\cdot,\lambda_n)\,|\,n\neq n_l,\ 1\leqq l\leqq N\}$ *is complete in* $L^2(a,b)$, *where* $(a,b)\subsetneqq(0,1)$.

§ 3. Outline of the proof of Theorems 2–4. Assume (1.4) and $0<x_0<1$. Suppose, for the moment, $N<\infty$ and

(3.1) $$(a,\phi(\cdot,\lambda_{n_l}))=0 \quad (0\leqq l\leqq N), \quad (a,\phi(\cdot,\lambda_n))\neq 0 \quad (n\neq n_l).$$

Put $\rho_n=\|\phi(\cdot,\lambda_n)\|^2_{L^2(0,1)}$, $\sigma_m=\|\psi(\cdot,\mu_m)\|^2_{L^2(0,1)}$ and $\mathcal{M}=\{(q,j,J)\,|\,$there exists some b such that $(q,j,J,b)\in\hat{\mathcal{M}}\}$. Then, the following lemma is obtained in the same way as in Suzuki [4] in virtue of Lemma 2. However, in deriving (3.2), Lemma 3 is made use of.

Lemma 4. *Under the assumption of* (3.1) *and* $0<x_0<1$, $(q,j,J)\in\mathcal{M}$ *if and only if there exists some* $K\in C^2(\bar{D})$ *such that* (2.4) *and*

(3.2) $$K(x_0,y)=K_x(x_0,y)=0 \qquad (0\leqq y\leqq x_0)$$

(3.3) $$\int_0^1 \{K_x(1,y)+JK(1,y)\}\phi(y,\lambda_n)dy=0 \qquad (n\neq n_l\,;\ 1\leqq l\leqq N)$$

(3.4) $$J=H-K(1,1).$$

Furthermore, the following facts hold: (I) *For each* $(q,j,J)\in\mathcal{M}$, *only a unique b satisfies* $(q,j,J,b)\in\hat{\mathcal{M}}$. (II) *Even if* $N=\infty$, *the if part of Lemma* 4 *holds under the assumption of the first part of* (3.1). (III) $(q,j,J)=(p,h,H)$ *if and only if* $K\equiv 0$ *on* $\bar{D}$. Therefore, Theorems 2–4 are proved by the following (A) and (B): (A) (2.4) *and* (3.2)–(3.4) *imply* $K\equiv 0$ *if* $1/2<x_0<1$ *and* $N<\infty$ *or if* $x_0=1/2$ *and* $N\leqq 1$. (B) *There exist* q, j, J and $K\not\equiv 0$ *such that* (2.4) and (3.2)–(3.4) *if* $0<x_0<1/2$ *or if* $x_0=1/2$ *and* $2\leqq N\leqq\infty$. Since the latter case is treated in a similar way to the former one in both (A) and (B), we only show (A) for the case of $1/2<x_0<1$ and $N<\infty$, and (B) for the case of $0<x_0<1/2$.

Set $D_{x_0}=D\cap\{(x,y)\,|\,x+y<2x_0\}$. By virtue of the uniqueness assertion of Propositions 1–3, (3.2) is equivalent to

(3.5) $$K=0 \qquad (\text{on } \overline{D_{x_0}})$$

under (2.4.a) and (2.4.c).

Proof of (A) *for the case of* $1/2<x_0<1$. By (3.5), we have $K(1,y)$

$=K_x(1, y)=0$ $(0 \leqq y \leqq 2x_0-1)$. Therefore, (3.7) gives

$$\int_{2x_0-1}^{1} \{K_x(1, y)+JK(1, y)\}\phi(y, \lambda_n)dy=0 \qquad (n \neq n_l, 1 \leqq l \leqq N),$$

hence

$$K_x(1, y)+JK(1, y)=0 \qquad (2x_0-1 \leqq y \leqq 1) \tag{3.6}$$

by Lemma 3. $K=0$ on $D \backslash D_{x_0}$ is derived from Proposition 3, by virtue of (2.4.a), (3.6) and (3.5).

Proof of (B) *for the case of* $0<x_0<1/2$. In this case (3.5) is equivalent to

$$K(x, 0)=0 \qquad (0 \leqq x \leqq 2x_0) \tag{3.7}$$

under (2.4.a) and (2.4.c), because of Proposition 2. Take an arbitrary $g \in C^2[0, 1]$ whose support is in $(2x_0, 1)$. In the same way as in Picard [2], we can show the unique existence of $K \in C^2(\bar{D})$ such that (2.4.a), (2.4.c), (3.6) and

$$K(x, 0)=g(x) \qquad (0 \leqq x \leqq 1). \tag{3.7$'$}$$

For the mapping

$$T=T_g : C^1[0, 1] \times \mathcal{R} \longrightarrow C^1[0, 1] \times \mathcal{R}$$
$$(q, J) \mid \rightarrow (2d/dxK(x, x)+p(x),\ H-K(1, 1)),$$

the following lemma is obtained by estimating each successive approximation of K:

Lemma 5. *There exist* $B>0$ *and* $\delta>0$ *such that* T_g *is a strict contraction mapping on* $U_B \equiv \{(q, J) \mid \|q\|_{C^1[0,1]}+|J| \leqq B\}$ *for each* $g \in C_0^2(2x_0, 1)$ *in* $\|g\|_{C^2[2x_0,1]} \leqq \delta$.

For $g \not\equiv 0$ with $\|g\|_{C^2[2x_0,1]} \leqq \delta$, there exists a fixed point of T_g, which is denoted by (q, J). Set $j=h+K(0, 0)$, $K \in C^2(\bar{D})$ being the solution of (2.4.a), (2.4.c), (3.6) and (3.7′). Then, q, j, J and K satisfy (2.4), (3.2)–(3.4), while $K \not\equiv 0$ follows from $g \not\equiv 0$.

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