11. *Fourier Coefficients of Siegel Cusp Forms of Degree Two*

By Yoshiyuki KITAOKA
Department of Mathematics, Nagoya University

(Communicated by Shokichi IYANAGA, M. J. A., Jan. 12, 1982)

Our aim is to estimate Fourier coefficients of Siegel cusp forms of degree two under an assumption about the estimates of generalized Kloosterman sums.

Let H be the space of 2×2 complex symmetric matrices whose imaginary part is positive definite and $\Gamma=Sp_2(Z)$. $\Lambda=\{S\in M_2(Z)\mid S={}^tS\}$, $\Lambda^*=\{S\in M_2(Q)\mid S=(s_{ij}),\ s_{ii}\in Z,\ 2s_{12}=2s_{21}\in Z\}$. $e(z)$ means $\exp(2\pi iz)$ for a complex number z.

Assumption. *Let $C\in M_2(Z)$, $|C|\neq 0$. For $G_1, G_2\in\Lambda^*$ we put*

$$K(G_1, G_2; C)=\sum_D e(\mathrm{tr}\,(AC^{-1}G_1+C^{-1}DG_2)),$$

where D runs over $\left\{D\in M_2(Z) \bmod C\Lambda \,\middle|\, \begin{pmatrix} * & * \\ C & D\end{pmatrix}\in\Gamma\right\}$ and $A\in M_2(Z)$ is any matrix such that $\begin{pmatrix} A & * \\ C & D\end{pmatrix}\in\Gamma$. For these generalized Kloosterman sums we assume for $0<\kappa\leq 1/2$,

$$K\left(G_1, G_2; \begin{pmatrix} c_1 & 0 \\ 0 & c_2\end{pmatrix}\right)=O(c_1^2c_2^{1-\kappa+\varepsilon}(c_2, g)^\kappa),$$

where $G_1, G_2\in\Lambda^*$, $c_1\mid c_2$ are natural numbers, ε is any positive number and g is the $(2,2)$-entry of G_2. ($\kappa=1/2$ is plausible.)

Theorem. *Let k be an even integer ≥ 6. Let*

$$f(Z)=\sum_{0<T\in\Lambda^*} a(T)e(\mathrm{tr}\,TZ)$$

be a cusp form of degree two, weight k. Suppose that Assumption is true, then we have

$$a(T)=O(|T|^{k/2-\kappa/2+\varepsilon}) \qquad \textit{for any } \varepsilon>0.$$

Sketch of the proof. Put $\Gamma_1=\left\{\begin{pmatrix} 1_2 & S \\ 0 & 1_2\end{pmatrix}\,\middle|\, S\in\Lambda\right\}$ and denote by $\mathfrak{h}$ the representatives of $\Gamma_1\backslash\Gamma/\Gamma_1$ and put $\theta(M)=\left\{S\in\Lambda \,\middle|\, M\begin{pmatrix} 1_2 & S \\ 0 & 1_2\end{pmatrix}M^{-1}\in\Gamma_1\right\}$ for $M\in\Gamma$. By virtue of [1], [4] we may assume

$$f(Z)=g(Z, Q)=\sum_{M\in\Gamma_1\backslash\Gamma} e(\mathrm{tr}\,(M\langle Z\rangle\cdot Q))|CZ+D|^{-k},$$

where $0<Q\in\Lambda^*$, $M=\begin{pmatrix} * & * \\ C & D\end{pmatrix}$. Then we have ([1])

$$g(Z, Q)=\sum_{M\in\mathfrak{h}}\sum_{0<T\in\Lambda^*} h(M, T)e(\mathrm{tr}\,TZ),$$

where

$$H(M,Z)=\sum_{S\in\Lambda/\theta(M)} e(\mathrm{tr}\,(M\langle Z+S\rangle\cdot Q))\,|C(Z+S)+D|^{-k},$$

$$h(M,T)=\int_{X \bmod 1} H(M,Z)e(-\mathrm{tr}\,TZ)dX, \qquad (X=\mathrm{Re}\,Z).$$

Let $0<T\in\Lambda^*$, $M=\begin{pmatrix} * & * \\ C & D\end{pmatrix}\in\mathfrak{h}$.

(i) If $C=0$, then $H(M,Z)=e(\mathrm{tr}\,(Q[U]\cdot Z))$, where $M=\begin{pmatrix} U & 0 \\ 0 & {}^tU^{-1}\end{pmatrix}$.

(ii) Suppose that $C=U^{-1}\begin{pmatrix} c_1 & 0 \\ 0 & 0\end{pmatrix}{}^tV$, $D=U^{-1}\begin{pmatrix} d_1 & d_2 \\ 0 & d_4\end{pmatrix}V^{-1}$ where U, $V\in GL_2(\boldsymbol{Z})$, c_1, $d_i\in \boldsymbol{Z}$, $c_1>0$, $d_4=\pm 1$. Put $Q[{}^tU]=\begin{pmatrix} p_1 & p_2 \\ p_2 & p_4\end{pmatrix}$, $T[{}^tV^{-1}]=\begin{pmatrix} s_1 & s_2 \\ s_2 & s_4\end{pmatrix}$ and take an integer a_1 such that $a_1d_1\equiv 1 \bmod c_1$. Then we have ([1])

$$\begin{aligned} h(M,T)=&\kappa_1\delta_{p_4,s_4}|T|^{k/2-3/4}p_4^{-1/2}c_1^{-3/2}e(-2s_2p_2d_4(c_1p_4)^{-1})\\ &\times e(\{a_1p_4d_2^2-2d_2(a_1p_2d_4-s_2)+a_1p_1+s_1d_1\}/c_1)\\ &\times J_{k-3/2}(4\pi\sqrt{|T||Q|}\,(s_4c_1)^{-1}),\end{aligned}$$

where δ is the Kronecker's delta function and κ_1 is a constant dependent on Q, and J is the usual Bessel function. Since $M\in\mathfrak{h}$ with $|C|=0$, $C\neq 0$ is parametrized by $U\in\left\{\begin{pmatrix} * & * \\ 0 & *\end{pmatrix}\in GL_2(\boldsymbol{Z})\right\}\Big\backslash GL_2(\boldsymbol{Z})$, $V\in GL_2(\boldsymbol{Z})\Big/\left\{\begin{pmatrix} 1 & * \\ 0 & *\end{pmatrix}\in GL_2(\boldsymbol{Z})\right\}$, $c_1\geq 1$, $d_4=\pm 1$, $d_i \bmod c_1$ $(i=1,2)$ with $(c_1,d_1)=1$, it is easy to see

$$\Bigg|\sum_{\substack{M\in\mathfrak{h}\\ |C|=0,\,C\neq 0}} h(M,T)\Bigg|\ll |T|^{k/2-1/4},$$

using $\sum_{x \bmod c} e((ax^2+bx)/c)=O((a,c)^{1/2}\cdot c^{1/2})$ and $J_{k-3/2}(x)=O(\min\,(x^{k-3/2}, x^{-1/2}))$.

(iii) Suppose $|C|\neq 0$. Decompose C as $C=U^{-1}\begin{pmatrix} c_1 & 0 \\ 0 & c_2\end{pmatrix}V^{-1}$, U, $V\in GL_2(\boldsymbol{Z})$, $c_i>0$, $c_1|c_2$. Then M in $\mathfrak{h}$ with $|C|\neq 0$ is parametrized by $c_1|c_2$, $U\in GL_2(\boldsymbol{Z})$, $V\in GL_2(\boldsymbol{Z})\Big/\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL_2(\boldsymbol{Z})\,|\, b\equiv 0 \bmod c_2/c_1\right\}$, $D \bmod C\Lambda$. We have ([1])

$$h(M,T)=|Q|^{-k/2+3/4}|T|^{k/2-3/4}|C|^{-3/2}e(\mathrm{tr}\,(QAC^{-1}+TC^{-1}D))\tilde{J}(\sqrt{T[\sqrt{Q}[{}^tC^{-1}]]}),$$

where $\tilde{J}(P)=\int_X e(-\mathrm{tr}\,(P(Z+Z^{-1})))\,|Z|^{-k}dX (Z=X+iY\in H)$. Moreover we have ([2])

$$\begin{aligned} h(M,T)=&\kappa_2|T|^{k/2-3/4}\,|C|^{-3/2}e(\mathrm{tr}\,(QAC^{-1}+TC^{-1}D))\\ &\times\int_0^1\prod_{i=1}^{2} J_{k-3/2}(4\pi s_it)t(1-t^2)^{-1/2}dt,\end{aligned}$$

where κ_2 is a constant depending on Q, and s_1, s_2 are eigenvalues of $\sqrt{T[\sqrt{Q}[{}^tC^{-1}]]}$. Since

$$\int_0^1 \prod_{i=1}^{2} J_{k-3/2}(4\pi s_i t)t(1-t^2)^{-1/2}dt \ll \begin{cases} |P|^{k/2-3/4} & \text{if } \operatorname{tr} P \ll 1, \\ |P|^{-1/4} & \text{if } \operatorname{tr} P \ll |P|, \\ |P|^{k/2-3/4}(\operatorname{tr} P)^{(1-k)/2} & \text{otherwise,} \end{cases}$$

where $P = T \cdot Q[{}^tC^{-1}]$, we have, under Assumption,

$$\sum_U \Big| \sum_{D \bmod CA} h(M, T) \Big| \ll |T|^{k/2-3/4} c_1^{1/2} c_2^{-1/2-\kappa+\varepsilon}((c_2, Q[v])^\kappa f(A),$$

where v is the second column of V and ε is any positive number and

$$f(A) = \sum_{\operatorname{tr} A[U] \ll 1} |A|^{k/2-3/4} + \sum_{\operatorname{tr} A[U] \ll |A|} |A|^{-1/4} + \sum_{\text{otherwise}} |A|^{k/2-3/4}(\operatorname{tr} A[U])^{(1-k)/2},$$

where $A = T\left[V\begin{pmatrix} c_1 & \\ & c_2 \end{pmatrix}^{-1}\right]$ and U runs over $GL_2(\boldsymbol{Z})$ with each condition. Since we have

$$f(A) \ll \begin{cases} m(A)^{(k-3)/2}|A|^{1/4} & \text{if } m(A)^{-1}|A| \gg \max(1, |A|), \\ m(A)^\varepsilon \max(1, |A|)^{(3-k)/2+\varepsilon} |A|^{k/2-5/4-\varepsilon} & \text{if } m(A)^{-1}|A| \ll \max(1, |A|), \end{cases}$$

where $m(A) = \min_{0 \neq x \in \boldsymbol{Z}^2} A[x]$ and ε is any positive number, we have

$$\sum_{U \in GL_2(\boldsymbol{Z})} \Big| \sum_{D \bmod CA} h(M, T) \Big| \ll |T|^{k/2-3/4} c_1^{1/2} c_2^{-1/2-\kappa+\varepsilon}(c_2, Q[v])^\kappa \times \begin{cases} m(A)^{(k-3)/2}|A|^{1/4} & \text{if } m(A)^{-1}|A| \gg \max(1, |A|), \\ m(A)^\varepsilon \max(1, |A|)^{(3-k)/2+\varepsilon} |A|^{k/2-5/4-\varepsilon} & \text{if } m(A)^{-1}|A| \ll \max(1, |A|). \end{cases}$$

Using Lemma 2 in [3], we complete the proof.

Details will appear elsewhere.

References

[1] U. Christian: Über Hilbert-Siegelsche Modulformen und Poincarésche Reihen. Math. Ann., **148**, 257–307 (1962).

[2] C. S. Herz: Bessel functions of matrix argument. Ann. of Math., **61**, 474–523 (1955).

[3] Y. Kitaoka: Modular forms of degree n and representation by quadratic forms. III. Kloosterman's method. Proc. Japan Acad., **57**A, 373–377 (1981).

[4] H. Maass: Über die Darstellung der Modulformen n-ten Grades durch Poincarésche Reihen. Math. Ann., **123**, 125–151 (1951).