99. *Characterization of Extremely Amenable Semigroups with a Unique Invariant Mean*

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§1. **Introduction.** Let S be an abstract semigroup and $m(S)$ the Banach space of all bounded real functions on S with the supremum norm. Let ψ be a mean on S i.e., ψ is a linear functional on $m(S)$ such that $\|\psi\|=\psi(I_S)=1$, where in general I_A is the characteristic function of any subset A of S. ψ is called *multiplicative* if $\psi(fg)=\psi(f)\psi(g)$ for any $f, g \in m(S)$. ψ is said to be *left* [*right*] *invariant* if $\psi(f)=\psi({}_sf)[\psi(f)=\psi(f_s)]$ for any $f \in m(S)$ and $s \in S$, where ${}_sf(t)=f(st)$ and $f_s(t)=f(ts)$ $(t \in S)$. By $LM(S)$ [$RM(S)$] we denote the set of all left [right] invariant means on S, and by $MLM(S)$ [$MRM(S)$] denote the set of all multiplicative left [right] invariant means on S. Any element in $LM(S) \cap RM(S)$ is called an *invariant mean* on S. We say that S is *left* [*right*] *amenable* if $LM(S)$ [$RM(S)$] $\neq \phi$ (empty). In particular S is called *extremely left* [*right*] *amenable* if $MLM(S)$ [$MRM(S)$] $\neq \phi$. Further S is called *extremely amenable* if $MLM(S) \cap MRM(S) \neq \phi$. It is proved in Granirer [2] that S is extremely left amenable if and only if it has the following property:

(CRZ) *For any $a, b \in S$ there exists $c \in S$ such that $ac=bc=c$.*
Suppose now S is extremely left amenable. Let us introduce a pseudo-order relation "$\leq$" in S defined as follows: For any $a, b \in S$, $a \leq b$ if either $a=b$ or $ab=b$. Then by virtue of (CRZ), $(S, \leq)$ is regarded as a directed set. A subset A of S is called *cofinal* (with respect to $\leq$) if for any given $s \in S$ there exists $t \in A$ such that $s \leq t$. For any fixed $p \in S$ we denote by δ_p a mean on S defined by $\delta_p(f)=f(p)$ $(f \in m(S))$.

The purpose of this paper is to prove the following theorems concerning semigroups with a unique multiplicative invariant mean.

Theorem 1. *Let S be extremely left amenable. The following conditions for S are equivalent:*

(1) *S admits a unique multiplicative left invariant mean.*

(2) *S admits a unique left invariant mean.*

(3) *For any subset A of S there exists $t \in S$ such that either $tS \subseteq A$ or $tS \cap A=\phi$.*

(4) *S has the property that if a subset A of S satisfies $sS \cap A \neq \phi$ for all $s \in S$, then $tS \subseteq A$ for some $t \in S$.*

(5) *Every cofinal subset of* $(S, \leq)$ *contains at least one right ideal in* S.

(6) *S has the zero element.*

Theorem 2. *The following conditions for S are equivalent:*

(1) *S admits a unique multiplicative left invariant mean.*

(2) *S admits a unique multiplicative right invariant mean.*

(3) *S admits a unique multiplicative invariant mean.*

(4) *S has the zero element.*

Theorem 3. *Let S be extremely left amenable and right amenable. If $MLM(S)$ is a finite set, then S has the zero element.*

§2. **Proof.** *Proof of Theorem* 1. As shown in Granirer [3, Theorem 6], if S is extremely left amenable, every extreme point of the w^*-compact convex subset $LM(S)$ in $m(S)^*$ is in $MLM(S)$. So we have (1)$\Rightarrow$(2). Suppose (2), and let $MLM(S)=LM(S)=\{\psi\}$. Then we see from Corollary 4 in Granirer [4, p. 68] that ψ must be given in the form

$$\psi(f)=\inf_{t\in S}\sup_{s\in S} f(ts)=\sup_{t\in S}\inf_{s\in S} f(ts) \qquad (f\in m(S)).$$

If $\psi(I_A)=1$ for $A\subseteq S$, $\sup_{t\in S}\inf_{s\in S} I_A(ts)=1$, which implies $tS\subseteq A$ for some $t\in S$. Since ψ is multiplicative, $\psi(I_A)=1$ or $\psi(I_S-I_A)=1$ for any subset A of S. Hence (2)$\Rightarrow$(3). (3)$\Rightarrow$(4) is obvious. Suppose (4), and let A be any cofinal subset of $(S, \leq)$ and $s\in S$. If $w\in sS\backslash A$, then there exists $t\in A$ with $w\leq t$. As $w\neq t$, we have $t=wt\in sS\cap A$. That is, $A\cap sS\neq\phi$ for all $s\in S$, and $tS\subseteq A$ for some $t\in S$ by (4). So (4)$\Rightarrow$(5). Suppose (5). Then we note that the family of all cofinal subsets in $(S, \leq)$ has the finite intersection property, because so has the family of all right ideals in S. If S does not contain a finite cofinal subset, it follows from Theorem of I. S. Luthar in Granirer [1, p. 372] that S contains at least two cofinal subsets which are disjoint. But this is impossible in our case. Hence there exists a finite cofinal subset A of $(S, \leq)$. Let us take $p\in A$ such that $a\leq p$ for all $a\in A$. Then the singleton $\{p\}$ is also cofinal, and $\{p\}=tS$ for some $t\in S$ by (5). Therefore p is the zero element in S. Thus (5)$\Rightarrow$(6). Finally (6)$\Rightarrow$(1) is evident.

Remark to Theorem 1. Suppose that S is extremely left amenable and satisfies (3) in Theorem 1. Let us consider a set function μ on the family of all subsets of S defined as follows: For any $A\subseteq S$, $\mu(A)=1$ if $tS\subseteq A$ for some $t\in S$, and $\mu(A)=0$ if $A\cap tS=\phi$ for some $t\in S$. Then we see that μ is a finitely additive probability measure on S and satisfies $\mu(A)=\mu(s^{-1}A)$ for any $A\subseteq S$ and $s\in S$, where $s^{-1}A=\{t\in S;\, st\in A\}$. Moreover the linear functional on $m(S)$ determined by μ is a unique multiplicative left invariant mean on S. Therefore (1) in Theorem 1 is derived directly from (3).

Proof of Theorem 2. Combining Theorem 1 with its right dual case, we see that (1), (2) and (4) in Theorem 2 are mutually equivalent. Suppose (3), and let $MLM(S)\cap MRM(S)=\{\psi\}$. By slight modification of discussions in [4], we conclude that ψ must be given in the form

$$\psi(f)=\inf_{t,u\in S}\sup_{s\in S} f(tsu)=\sup_{t,u\in S}\inf_{s\in S} f(tsu) \qquad (f\in m(S)).$$

On the other hand, since S is extremely amenable, it is regarded as a directed set with respect to a partial order relation "(r)" in S defined as follows: For any a, b in S, $a(r)b$ if either $a=b$ or $ab=ba=b$. Making use of these facts, we see by the same way as in the proof of Theorem 1 that S has the zero element.

Proof of Theorem 3. Let $\psi\in RM(S)$. If $MLM(S)$ is a finite set, it follows from Theorem 1 in [1] that S has at least one right zero p. Since $f_p=f(p)I_S$ for $f\in m(S)$, we have $\psi(f)=\psi(f_p)=f(p)\psi(I_S)=f(p)$ and $f(ps)=f_s(p)=\psi(f_s)=\psi(f)=f(p)$ for any $f\in m(S)$ and $s\in S$. Hence p is the zero in S.

Remarks to Theorem 3. 1) In Theorem 3 we can not drop the assumption that S is right amenable. For example let S be a right zero semigroup of finite order $n\geq 2$. Then $MLM(S)$ consists exactly of n elements, and $RM(S)=\phi$.

2) As consequences of Theorem 3 and Theorem 2 in [1], we have the following assertions:

(a) *$MLM(S)$ consists of n elements $(1\leq n<\infty)$ if and only if the set $RZ(S)$ of all right zeros in S contains exactly n elements. In this case $MLM(S)=\{\delta_p\,;\,p\in RZ(S)\}$.*

(b) *If $MLM(S)$ is a finite set containing at least two elements, then S is not right amenable.*

§3. A generalization of Theorem 1. Let (S, X) be a left action of S on a space X, and $m(X)$ the Banach space of all bounded real functions on X with the supremum norm. A mean ψ on X (cf. §1) is called *S-invariant* if $\psi({}_sf)=\psi(f)$ for any $f\in m(X)$ and $s\in S$, where ${}_sf(x)=f(sx)(x\in X)$. We say that (S, X) is [*extremely*] *S-amenable* if there exists a [multiplicative] S-invariant mean on X. As noted in [6, Theorem 2.6], (S, X) is [extremely] S-amenable if S is [extremely] left amenable. If (S, X) is S-amenable, then the family $\{sX\,;\,s\in S\}$ of subsets in X has the finite intersection property. Further if S is extremely left amenable, every extreme point of the w^*-compact convex subset of all S-invariant means in $m(X)^*$ is multiplicative (cf. [5, p. 427]). Making use of these facts and results stated in [7, §§3–4], we have the following theorem, which is a natural generalization of Theorem 1.

Theorem 4. *Let S be extremely left amenable, and (S, X) be a left action of S on a space X. The following conditions for (S, X) are equivalent:*

(1) *(S, X) admits a unique multiplicative S-invariant mean.*

(2) *(S, X) admits a unique S-invariant mean.*

(3) *For any subset Y of X there exists $s \in S$ such that either $sX \subseteq Y$ or $sX \cap Y = \phi$.*

(4) *(S, X) has the property that if a subset Y of X satisfies $sX \cap Y \neq \phi$ for all $s \in S$, then $tX \subseteq Y$ for some $t \in S$.*

Moreover assume that $\cap\{sX; s \in S\}$ is nonempty. Then the above conditions are equivalent to:

(5) *There exists a unique point $p \in X$ such that $p = sp = tX$ for all $s \in S$ and some $t \in S$.*

References

[1] E. E. Granirer: A theorem on amenable semigroups. Trans. Amer. Math. Soc., **111**, 367–379 (1964).
[2] ——: Extremely amenable semigroups. Math. Scand., **17**, 177–197 (1965).
[3] ——: ditto. II. ibid., **20**, 93–113 (1967).
[4] ——: Functional analytic properties of extremely amenable semigroups. Trans. Amer. Math. Soc., **137**, 53–75 (1969).
[5] K. Sakai: Extremely amenable transformation semigroups. Proc. Japan Acad., **49**, 424–427 (1973).
[6] ——: On amenable transformation semigroups I. J. Math. Kyoto Univ., **16**, 555–595 (1976).
[7] ——: ditto. IV. Sci. Rep. Kagoshima Univ., **31**, 1–19 (1982).