95. Perturbations of Compact Foliations

By Kazuhiko FUKUI
Department of Mathematics, Kyoto Sangyo University

(Communicated by Heisuke HIRONAKA, M. J. A., Oct. 12, 1982)

1. Introduction. A compact foliation F is one in which every leaf is compact. The problem we wish to consider concerns foliations F' whose plane fields are close, in some C^r-topology, to the plane field tangent to the leaves of F. Such an F' is called a C^r-perturbation of F. Then, the following question arises: When does F' have compact leaf? The first result of this nature is due to H. Seifert [7]. He proved that any C^0-perturbation of any orientable S^1-bundle over a surface B of $\chi(B)\neq 0$ has a compact leaf, where $\chi(B)$ is the euler characteristic number of B. R. Langevin and H. Rosenberg [5] considered a fibration $q: E\to B$ with fibre L, B a closed surface, E closed. And they proved that any C^0-perturbation of this fibration has a compact leaf provided that $\pi_1(L)\cong Z$, B is a surface with $\chi(B)\neq 0$ and $\pi_1(B)$ acts trivially on $\pi_1(L)$.

The purpose of this note is to investigate some properties of perturbations of compact codimension two foliations and generalize the above result.

2. Preliminaries and statement of results. Let M be a compact manifold without boundary and F a compact codimension two foliation. Then by the results of D. B. A. Epstein [2] and R. Edwards, K. Millett and D. Sullivan [1], we have the following: There is an upper bound on the volumes of the leaves of F. There is an equivalent formulation as follows.

Proposition 1 (D. B. A. Epstein [3]). *There is a generic leaf L_0 with property that there is an open dense subset of M, where the leaves have all trivial holonomy and are all diffeomorphic to L_0. Given a leaf L, we can describe a neighborhood $U(L)$ of L, together with the foliation on the neighborhood as follows. There is a finite group $G(L)$ of $O(2)$. $G(L)$ acts freely on L_0 on the right and $L_0/G(L)\cong L$. Let D^2 be the unit disk. We foliate $L_0\times D^2$ with leaves of the form $L_0\times\{pt\}$. This foliation is preserved by the diagonal action of $G(L)$, defined by $g(x,y)=(x\cdot g^{-1}, g\cdot y)$ for $g\in G(L)$, $x\in L_0$ and $y\in D^2$. So we have a foliation induced on $U=L_0\times_{G(L)}D^2$. The leaf corresponding to $y=0\in D^2$ is $L_0/G(L)$. Then there is a C^∞-imbedding $\varphi: U\to M$ with $\varphi(U)=U(L)$, which preserves leaves and $\varphi(L_0/G(L))=L$.*

Definition 2. A leaf L is called *singular* if $G(L)$ is not trivial. The order of $G(L)$ is called the order of holonomy of L.

Definition 3. A singular leaf L is called *isolated* if the action of $G(L)$ has only the origin of D^2 as fixed point.

We assume that the fundamental groups of the leaves of F are all isomorphic to Z.

From Proposition 1, we see that each isolated singular leaf is isolated, hence there are finitely many isolated singular leaves in F because that M is compact. And the set S of non-isolated singular leaves of F is a submanifold of M. The leaf space M/F denoting by B is a compact V-manifold of dimension two and the quotient map $q: M \to B$ is a V-bundle (for definitions, see I. Satake [6]). In this case, $q(S)$ is the boundary of B if S is non-empty. Let $L_1, \cdots, L_n$ denote all the isolated singular leaves of F with holonomy of order $k_1, \cdots, k_n$ respectively. Put $p_i = q(L_i)$ $(i=1, \cdots, n)$ and $\partial B = q(S)$. Note that the restriction $q: M - S \cup \{L_1, \cdots, L_n\} \to M - S \cup \{L_1, \cdots, L_n\}/F = B - \partial B \cup \{p_1, \cdots, p_n\}$ is a fibration with a generic leaf L as fibre. Thus $\pi_1(B - \partial B \cup \{p_1, \cdots, p_n\})$ acts on $\pi_1(L)$. We see that each $G(L_i)$ is isomorphic to a finite cyclic group and $G(L)(L \in S)$ is isomorphic to Z_2 whose generator is a reflection. By Proposition 1, for each isolated singular leaf L_i, the restriction $q: \partial U(L_i) \cong L \times_{G(L_i)} \partial D^2 \to \partial D^2/G(L_i)$ is a fibration with fibre L. Then we can see that $\pi_1(\partial D^2/G(L_i)) \cong Z$ acts trivially on $\pi_1(L)$. Moreover, since the inclusion $B - \partial B \to B$ is a homotopy equivalence, we may consider that $\pi_1(B)$ also acts on $\pi_1(L)$.

We let F' be a sufficiently small perturbation of F. Then by the result of M. W. Hirsch ([4], Theorem 1.1), we have the following: For each $U(L_i)$, $F'|_{U(L_i)}$ has a compact leaf L_i' in $U(L_i)$ such that there is a diffeomorphism $h_i: L_i \to L_i'$. We remark that F' has at least n compact leaves. Let α be a loop in a generic leaf L representing a generator of $\pi_1(L)$ and α_i (resp. $\alpha_i' = h_i(\alpha_i)$) a loop in L_i (resp. L_i') representing a generator of $\pi_1(L_i)$ (resp. $\pi_1(L_i')$) such that $j_i(\alpha) = k_i \alpha_i$, where $j_i: L \to L_i$ is a canonical projection. Let $H(\alpha_i)$ (resp. $H(\alpha_i')$) be the holonomy map of α_i (resp. α_i') for F (resp. F'), which is a local diffeomorphism of $(R^2, 0)$. Thus, if $H(\alpha_i)$ has no fixed point except for the origin 0, we can define the fixed point index of $H(\alpha_i)$ at 0 in the usual way. We denote it by $I(H(\alpha_i), L_i)$. Now we are in a position to state our theorem.

Theorem 4. *Let M be a compact smooth manifold without boundary and F a compact codimension two foliation of M with leaf space B. We assume that $\pi_1(L) \cong Z$ for each leaf L of F and $\pi_1(B)$ acts trivially on $\pi_1(L)$. Let F' be a foliation of M, C^0-close to F. If F' has exactly n compact leaves, then we have a following relation:*

$$\chi(B) + \sum_{i=1}^{n} \left(\frac{1}{k_i} - 1 \right) = \sum_{i=1}^{n} \frac{1}{k_i} I(H(\alpha_i')^{k_i}, L_i').$$

Corollary 5. *Let M be a compact manifold without boundary and F a compact codimension two foliation of M which has no isolated singular leaves. Suppose that*

1) $\pi_1(L) \cong Z$ *for each leaf L of F,*
2) $\pi_1(B)$ *acts trivially on* $\pi_1(L)$ *and*
3) $\chi(B) \neq 0$.

Then any C^0-perturbation of F has a compact leaf.

This result is an extension of the results of Seifert [7] and Langevin and Rosenberg [5].

Example 6. The Klein bottle K^2 is an S^1-bundle over S^1 with structure group Z_2. Then we can construct a compact foliation G of K^2 such that G is transverse to the fibres and has two isolated singular leaves. We foliate $S^1 \times K^2$ with leaves of the form $\{pt\} \times L$, $L \in G$. This foliation F is a compact codimension two foliation with no isolated singular leaves and the leaf space $S^1 \times K^2/F$ is homeomorphic to a cylinder $S^1 \times [0, 1]$. Thus the euler characteristic number of $S^1 \times K^2/F$ is equal to zero. Furthermore there exists a C^0-perturbation F' of F such that F' has no compact leaves. This example shows that the condition 3) of Corollary 5 is essential.

Corollary 7. *Under the assumption of Theorem 4, we suppose that each $H(\alpha_i')$ is expanding or contracting. If $\chi(B) \neq n$, then F' has at least $n+1$ compact leaves.*

Corollary 8. *Under the assumption of Theorem 4, we suppose that 1 is not an eigenvalue of $LH(\alpha_i')^{k_i} \in GL(2, R)$ for each i, where $LH(\alpha_i')$ is the linear holonomy of $H(\alpha_i')$. If $\chi(B) < 0$, then F' has at least $n+1$ compact leaves.*

3. Sketch of the proofs. The detailed proofs of the above results will appear elsewhere. We give here only an outline. We let F' be a C^0-perturbation of F. At the first step we construct a generalized first return map $f: M \to M$ associating to F' and a vector field X from the map f, whose zero's give compact leaves of F' (cf. § 1 of [5]). Let D be a 2-disk in B. At the second step we construct a compact 2-submanifold B^* of M over B- int (D), such that B^* is transverse to F and the map $q: B^* \to B$- int (D) is a V-covering. At the third step we construct a closed 2-manifold $\bar{B}^*$ by pasting some 2-disks to B^* along ∂B^*. The vector field X projects a vector field X^* tangent to B^*. And we extend the vector field X^* to $\bar{B}^*$. We have Theorem 4 by computing the euler characteristic number of $\bar{B}^*$. Corollaries 5, 7 and 8 are proved by using Theorem 4.

References

[1] R. Edwards, K. Millet, and D. Sullivan: Foliations with all leaves compact. Topology, **16**, 13–32 (1977).

[2] D. B. A. Epstein: Periodic flows on three-manifolds. Ann. of Math., **95**, 68–82 (1976).

[3] ——: Foliations with all leaves compact. Ann. Inst. Fourier, **26**, 265–282 (1976).

[4] M. W. Hirsch: Stability of compact leaves of foliations. Dynamical Systems. Academic Press, pp. 135–155 (1971).

[5] R. Langevin and H. Rosenberg: Integrable perturbations of fibrations and a theorem of Seifert, Differential topology, foliations and Gelfand-Fuks cohomology. Lect. Notes in Math., vol. 652, pp. 122–127 (1978).

[6] I. Satake: The Gauss-Bonnet theorem for V-manifolds. J. Math. Soc. Japan, **9-4**, 464–492 (1917).

[7] H. Seifert: Closed integral curves in 3-space and isotopic two dimensional deformations. Proc. Amer. Math. Soc., **1**, 287–302 (1950).