73. 2-Dimensional Periodic Continued Fractions and 3-Dimensional Cusp Singularities

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2-dimensional cusp singularities are in one-to-one correspondence with periodic continued fractions, which may be interpreted as cycles of integers. We regard a cycle of integers, as a triangulation of a circle on each vertex of which an integer is attached. Then as a generalization of a periodic continued fraction to dimension 2, we consider a triangulation of a compact topological surface on each edge of which a pair of integers is attached. We show that if it satisfies some conditions, then it induces a 3-dimensional cusp singularity in a manner similar to the 2-dimensional case. Then the singularity has a resolution whose exceptional set is completely determined by the given triangulation realized as the "dual graph". The cusp singularities thus obtained have a duality among themselves generalizing that of Nakamura [2]. In the special case of real tori, we get Hilbert modular cusp singularities.

The details will appear elsewhere.

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Results. Let $N=\boldsymbol{Z}^n$ and $N_{\boldsymbol{R}}=N\otimes_{\boldsymbol{Z}}\boldsymbol{R}\simeq\boldsymbol{R}^n$. Let $\pi: N_{\boldsymbol{R}}\backslash\{0\}\to S^{n-1}$ be the natural projection onto a sphere $S^{n-1}=(N_{\boldsymbol{R}}\backslash\{0\})/\boldsymbol{R}_{>0}$. Then $\mathrm{Aut}(N)=GL(N)$ acts on S^{n-1} through π. Let $\mathcal{S}$ be the set of the pairs (C,Γ) of a cone C in $N_{\boldsymbol{R}}$ and a subgroup Γ of $GL(N)$ satisfying the following conditions: C is open, nondegenerate (i.e., $\overline{C}\cap(\overline{-C})=\{0\}$), convex and Γ-invariant. Moreover, the induced action of Γ on $D=\pi(C)=C/\boldsymbol{R}_{>0}$ is properly discontinuous and fixed point free with the compact quotient D/Γ.

Let $T_N=N\otimes_{\boldsymbol{Z}}\boldsymbol{C}^*\simeq(\boldsymbol{C}^*)^n$ and let $\mathrm{ord}=-\log|\ \ |: T_N\to N_{\boldsymbol{R}}=T_N/CT_N$ be the canonical map, where CT_N is the compact real torus $N\otimes_{\boldsymbol{Z}}U(1)\simeq U(1)^n$. Using the theory of torus embeddings [2] we can show the following:

Theorem 1. *If (C,Γ) is in $\mathcal{S}$, then we have an n-dimensional cusp singularity $(V,p)=\mathrm{Cusp}\,(C,\Gamma)$ such that $V\backslash\{p\}\simeq\mathrm{ord}^{-1}(C)/\Gamma$.*

Let $\mathcal{T}=\{\mathrm{Cusp}\,(C,\Gamma)\,|\,(C,\Gamma)\in\mathcal{S}\}$. We have a duality in $\mathcal{T}$ in the following way: Let C^* be the dual cone of C in the dual vector space $M_{\boldsymbol{R}}=N^*_{\boldsymbol{R}}$ of $N_{\boldsymbol{R}}$. Then Γ also acts on M and C^* canonically and (C^*,Γ)

is in $\mathcal{S}$. We call Cusp (C^*, Γ) the dual singularity of Cusp (C, Γ).

The well-known Hilbert modular cusp singularities are contained in $\mathcal{T}$. For a totally real algebraic number field K of degree n over $\boldsymbol{Q}$, C is the totally positive orthant in $\boldsymbol{R}\otimes_{\boldsymbol{Q}} K$ and Γ in a group of totally positive units of rank $n-1$. D/Γ in this case is an $(n-1)$-dimensional real torus.

Next, we explain how to construct (C, Γ) in $\mathcal{S}$ systematically when $n=3$, generalizing the notion of periodic continued fractions for $n=2$. In the following, we use the notations of Oda [2]. Let T be a compact topological surface, let $\tilde{T}\to T$ be its universal covering space and let $\Gamma=\pi_1(T)$, the fundamental group of T. Let Δ be a Γ-invariant triangulation of $\tilde{T}$.

Definition 2. A Γ-invariant double $\boldsymbol{Z}$-weighting of Δ satisfying the *monodromy condition* at the vertices is a pair of integers attached to each edge of Δ with one integer on the side of one vertex and with the other integer on the side of the other vertex satisfying the following conditions: (i) These integers are Γ-invariantly attached. (ii) For each vertex v of Δ, let $v_1, v_2, \cdots, v_s$ be the vertices of its link going around v in this order. Let $\{n_1, n_2, n\}$ be an arbitrary $\boldsymbol{Z}$-basis of N. Then we can determine $n_3, \cdots, n_s$ and n_{s+1} in N by the equality (*) $n_{j-1}+n_{j+1}+a_j n_j+b_j n=0$, where (a_j, b_j) is the given pair of integers on the edge joining v_j and v with a_j (resp. b_j) on the side of v_j (resp. v). Then we require that $n_{s+1}=n_1$ and that their images $\pi(n_1), \pi(n_2), \cdots, \pi(n_s)$ in the sphere S^2 go around $\pi(n)$ exactly once in this order.

Let Δ be a Γ-invariant triangulation of $\tilde{T}$, endowed with a Γ-invariant double $\boldsymbol{Z}$-weighting satisfying the monodromy condition at the vertices. Choose and fix a $\boldsymbol{Z}$-basis $\{n, n', n''\}$ and a triangle of Δ with vertices $\{v, v', v''\}$. Then, since $\tilde{T}$ is simply connected, we get the N-weighting map σ: {all vertices of $\Delta\}\to N$ which sends v, v', v'' to n, n', n'', respectively, and which sends other vertices to the elements of N determined by the equality (*) above. Moreover, we have a unique homomorphism $\rho: \Gamma\to GL(N)$ satisfying $\rho(\gamma)\cdot\sigma(v)=\sigma(\gamma\cdot v)$ for any element γ of Γ and any vertex v of Δ. We easily obtain a Γ-equivariant local homeomorphism $f: \tilde{T}\to S^2$, extending the map $\pi\cdot\sigma$ such that the image of each triangle of Δ is a spherical triangle.

Theorem 3. *Assume that the following condition (**) is satisfied:*

(**) *f is injective, $f(\tilde{T})$ is spherically convex and its closure $\overline{f(\tilde{T})}$ is contained in a hemisphere of S^2.*

Then $(C, \rho(\Gamma))$ is in $\mathcal{S}$, where $C=\pi^{-1}(f(\tilde{T}))$.

By this theorem, we have a cusp singularity Cusp $(C, \rho(\Gamma))$. Then it has a resolution whose exceptional set consists of rational surfaces crossing each other along rational curves and points in such a way that the "dual graph" agrees with the given triangulation Δ. We have the following sufficient condition, under which (**) is satisfied.

Theorem 4. *Let Δ be a Γ-invariant triangulation of the universal covering space $\tilde{T}$ of a compact topological surface T, endowed with a Γ-invariant double $\boldsymbol{Z}$-weighting satisfying the monodromy condition at the vertices, where $\Gamma=\pi_1(T)$ is the fundamental group of T. Then the map $f: \tilde{T}\to S^2$ induced by Δ, as above, satisfies the condition* (**) *of Theorem* 3, *if the following two conditions are satisfied*: (i) *The sum of the double $\boldsymbol{Z}$-weights on each edge of Δ is not greater than -2.* (ii) *We get a cell division of $\tilde{T}$ by deleting all the edges of Δ which have the sum of the double $\boldsymbol{Z}$-weights equal to -2.*

An example. Let Δ_1 be an octahedral triangulation of a 2-sphere S^2. Take a double covering T of S^2 ramifying at all six vertices of Δ_1 and let Δ_2 be the triangulation of T induced by Δ_1. Then T is a compact orientable surface of genus 2. Let Δ be the triangulation of the universal covering space $\tilde{T}$ of T induced by Δ_2, and let $\Gamma=\pi_1(T)$. We have a Γ-invariant double $\boldsymbol{Z}$-weighting of Δ satisfying the monodromy condition at the vertices if we attach integers on each triangle of Δ, as in Fig. 1. Clearly, it satisfies the conditions of Theorem 4.

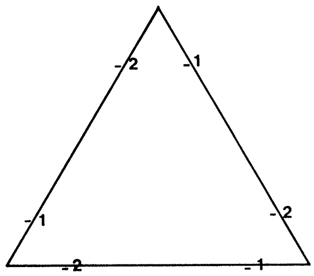

Fig. 1

References

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[2] T. Oda: Lectures on torus embeddings and applications (Based on joint work with K. Miyake). Tata Inst. of Fund. Res., 1978.