7. On Laplacian and Hessian Comparison Theorems

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(Communicated by Kunihiko KODAIRA, M. J. A., Jan. 12, 1982)

In this paper, we shall state Laplacian and Hessian comparison theorems and give their applications to the following subjects:

(1) the theory of harmonic functions on a Riemannian manifold,

(2) the geometric structure of a Riemannian manifold with boundary.

The full proofs of the results in this paper will be given in the forthcoming papers [6] and [7].

Throughout this paper, let M be a connected Riemannian manifold of dimension m with (possibly empty) smooth boundary ∂M. We write M_0 for the interior of M and M_x for the tangent space at x. Let $\operatorname{dis}(x, y)$ denote the distance between two points x and y defined by the Riemannian metric g of M.

§1. Let N be a closed subset of M and x a point of $M_0 \backslash N$. Suppose there is a geodesic $\sigma: [0, l] \rightarrow M$ such that $\sigma((0, l]) \subset M_0$, $\sigma(l) = x$ and $\operatorname{dis}(N, \sigma(t)) = t$ for $t \in [0, l]$. We choose two continuous functions R and K on $[0, l]$ such that

(1.1) the Ricci curvature in direction $\sigma(t) \geqq (m-1)R(t)$,

(1.2) the sectional curvature of any plane containing $\dot{\sigma}(t) \leqq K(t)$.

When N is a closed submanifold of dimension n, we choose a real number Λ such that

(1.3) the trace of $S_{\dot{\sigma}(0)} \leqq n\Lambda$,

where $S_{\dot{\sigma}(0)}$ is the second fundamental form of N with respect to $\dot{\sigma}(0)$ (i.e., $g(S_{\dot{\sigma}(0)}X, Y) = g(\nabla_X \dot{\sigma}(0), Y)$). Let f, h, and H be, respectively, the solutions of the equations:

(1.4) $f'' + Rf = 0$ with $f(0) = 0$ and $f'(0) = 1$,

(1.5) $h'' + Rh = 0$ with $h(0) = 1$ and $h'(0) = \Lambda$,

(1.6) $H'' + KH = 0$ with $H(0) = 1$ and $H'(0) = 0$.

With these preparations, we have the following theorems.

Theorem 1 (Laplacian comparison theorem). *For any nondecreasing C^2-function ψ on $(0, l]$, the distance function $\rho = \operatorname{dis}(N, *)$ satisfies*

(1.7) $\Delta\psi(\rho)(x) \leqq (\psi'' + (m-1)\psi' f'/f)(\rho(x))$.

In the case when N is a hypersurface, we have

(1.8) $\Delta\psi(\rho)(x) \leqq (\psi'' + (m-1)\psi' h'/h)(\rho(x))$.

Similarly, we can obtain upper estimates of the Hessian $\nabla^2\psi(\rho)$ in terms of the lower bounds of the sectional curvature along σ and moreover the upper bound of the eigenvalues of $S_{\dot\sigma(0)}$ when N is a submanifold. As for the lower estimate of $\nabla^2\psi(\rho)$, we have the following

Theorem 2 (Hessian comparison theorem). *We assume M is moreover a complete Riemannian manifold without boundary. Suppose the sectional curvature of M is nonpositive and N is a totally convex closed subset of M. Then we have*

$$\nabla^2\psi(\rho)_x(X,X)\geqq\psi''(\rho(x))g(\dot\sigma(l),X)^2+(\psi'H'/H)(\rho(x))\{g(X,X)-g(\dot\sigma(l),X)^2\}$$

for any $X\in M_x$.

Here we call a closed subset N *totally convex* if for any $x,y\in N$ and every geodesic $\gamma:[0,1]\to M$ $(\partial M=\phi)$ such that $\gamma(0)=x$ and $\gamma(1)=y$, the image of γ is contained in N.

We remark that when σ can be extended to a geodesic $\tilde\sigma:[0,\tilde l]\to M$ $(l<\tilde l)$ such that $\operatorname{dis}(N,\tilde\sigma(t))=t$ for $t\in[0,\tilde l]$, inequality (1.7) is exactly the Laplacian comparison theorem of Greene and Wu [2] in the case N is a point and inequality (1.8) is implicit in a comparison theorem by Heintze and Karcher [3]. However in our situations, Theorem 1 does not follow from the known comparison theorems such as mentioned above. In fact, our proof of Theorem 1 requires much more delicate arguments (cf. Wu [9]).

§2. In this section, we assume M is a complete, noncompact Riemannian manifold without boundary. We call M *hyperbolic* if it has a nonconstant positive superharmonic function and *parabolic* if it is not hyperbolic. Let Ω be a compact domain in M such that the boundary N is smooth. Let W_Ω be the lower envelope of the family of nonnegative superharmonic functions on M which dominate 1 on Ω. Then W_Ω is called the *harmonic measure* of Ω relative to M. It is well known that M is parabolic if $W_\Omega=1$ on M and hyperbolic if $W_\Omega\not\equiv 1$ on M (cf. e.g. [1, p. 135]). Now we choose a continuous function R on $[0,\infty)$ and a real number Λ such that for any geodesic $\sigma:[0,l]\to M$ with $\operatorname{dis}(N,\sigma(t))=t$ $(t\in[0,l])$, R satisfies inequality (1.1) and Λ satisfies inequality (1.3). Let h be the solution of equation (1.5) defined by R and Λ. Set

$$\Psi_r(t)=1/C_r\int_t^r 1/h^{m-1}\ (r>0)\quad\text{and}\quad \Psi(t)=\lim_{r\to\infty}\Psi_r(t),$$

where

$$C_r=\int_0^r 1/h^{m-1}.$$

Then by inequality (1.8), we see that $\Psi_r(\rho_\Omega)$ $(\rho_\Omega=\operatorname{dis}(\Omega,*))$ is subharmonic on $M\backslash\Omega$, $\Psi_r(\rho_\Omega)=1$ on N and $\Psi_r(\rho_\Omega)=0$ on $\{x\in M:\rho_\Omega(x)=r\}$. Therefore we see that for each $r>0$, $W_\Omega\geqq\Psi_r(\rho_\Omega)$ on $M\backslash\Omega$. Thus we obtain the following

Theorem 3. *$W_\Omega\geqq\Psi(\rho_\Omega)$ on $M\backslash\Omega$. In particular, if*

$$\int_0^\infty 1/h^{m-1}=\infty,$$

then M is parabolic.

In the rest of this section, we assume M has nonpositive sectional curvature and contains a totally convex closed subset Ω. Then we have the following

Theorem 4. *Suppose there is a nonpositive continuous function K on $[0, \infty)$ such that the sectional curvature at $x \in M$ is bounded from above by $K(\rho_\Omega(x))$ ($\rho_\Omega=\mathrm{dis}(\Omega, *)$), and moreover K is not identically zero if $m \geqq 3$ or K satisfies $K(t) \leqq -(1+\varepsilon)/(t^2 \log t)$ on $[a, \infty)$ for some $\varepsilon>0$ and $a>0$ if $m=2$. Then there is a superharmonic function Φ such that $0<\Phi\leqq 1$ on M, $\Phi=1$ on Ω, and $\Phi(x)$ tends to 0 as $\rho_\Omega(x)\to\infty$.*

Proof. Let H be the solution of equation (1.6) defined by K. Then by the assumptions on K, we see that

$$\int_0^\infty 1/H^{m-1}<+\infty$$

(cf. [8] in the case when $m=2$). We now define a continuous function on M by $\Phi=1/C\int_{\rho_\Omega}^\infty 1/H^{m-1}$ on $M\backslash\Omega$ and $\Phi=1$ on Ω, where

$$C=\int_0^\infty 1/H^{m-1}.$$

Then by Theorem 2, we see that Φ is a required function.

As an application of Theorem 4, we have the following

Corollary. *Let M and Ω be as in Theorem 4. Suppose Ω separates M. Then M has a nonconstant bounded harmonic function. Moreover if $M\backslash\Omega$ has a connected component whose boundary is compact, then there is a nonconstant harmonic function with finite Dirichlet norm.*

We remark that Theorem 3 contains as a special case a theorem of Ichihara ([5, Theorem 2.1]) and the first assertion of Corollary is a generalization of a result by Greene and Wu ([2, Proposition 7.1]). Moreover we notice that each of the conditions in Theorems 3, 4 and Corollary is optimum.

§ 3. In this section, we assume M is complete and the boundary ∂M is smooth. We say M is *of class* (R, Λ) ($R, \Lambda \in \boldsymbol{R}$) if the Ricci curvature of M is bounded from below by $(m-1)R$ and the trace of S_ξ is bounded from above by $(m-1)\Lambda$, where S_ξ is the second fundamental form of ∂M with respect to the unit inner normal vector field ξ on ∂M. Set $i(M)=\sup\{\mathrm{dis}(x, \partial M): x \in M\}$ ($\leqq+\infty$), $C_1(R, \Lambda)=\inf\{t: t>0, h(t)=0\}$ ($\leqq+\infty$) and $C_2(R, \Lambda)=\inf\{t: t>0, h'(t)=0\}$ ($\leqq+\infty$), where h is the solution of the equation (1.5) defined by R and Λ. Then as applications of Theorem 1, we have the following theorems.

Theorem 5. *Let M be a Riemannian manifold of class (R, Λ). Then:*

(1) $i(M) \leqq C_1(R, \Lambda)$.

(2) *If* $C_1(R, \Lambda) < +\infty$ *and* $\operatorname{dis}(p, \partial M) = C_1(R, \Lambda)$ *for some* $p \in M$, *then* M *is isometric to the closed metric ball with radius* $C_1(R, \Lambda)$ *in the simply connected space form of constant sectional curvature* R.

(3) $C_1(R, \Lambda) < +\infty$ *if and only if* $R > 0$, $R = 0$ *and* $\Lambda < 0$, *or* $R < 0$ *and* $\Lambda < -\sqrt{-R}$.

Theorem 6. *Let* M *be a Riemannian manifold of class* (R, Λ). *Suppose* ∂M *is disconnected and it has a compact connected component, say* Γ_1. *Then*:

(1) *If* $R = 0$ *and* $\Lambda = 0$, M *is the isometric product* $[0, a] \times \Gamma_1$.

(2) *If* $R > 0$, *then* $\Lambda > 0$ *and* $\min_{2 \leqq j} \operatorname{dis}(\Gamma_1, \Gamma_j) \leqq 2C_2(R, \Lambda)$, *where* $\{\Gamma_j\}_{j=1,2,\dots}$ *are the connected components of* ∂M. *Moreover if* $\min_{2 \leqq j} \operatorname{dis}(\Gamma_1, \Gamma_j) = 2C_2(R, \Lambda)$, *then* M *is isometric to the warped product* $[0, 2C_2(R, \Lambda)] \times_h \Gamma_1$.

Theorem 7. *Let* M *be a Riemannian manifold of class* (R, Λ). *Suppose* ∂M *is compact but* M *is noncompact. Then*:

(1) $R \leqq 0$.

(2) *If* $R = 0$ *and* $\Lambda = 0$, *then* ∂M *is connected and* M *is the isometric product* $[0, \infty) \times M$.

(3) *If* $\Lambda < 0$, *then* $R < 0$ *and* $\Lambda \geqq -\sqrt{-R}$. *Moreover if* $\Lambda = -\sqrt{-R}$, *then* M *is isometric to the warped product* $[0, \infty) \times_h M$.

After the preparation of [7], the author was informed that Ichida [4] has also shown the assertion (1) of Theorem 6, independently.

Finally the author would like to express sincere thanks to Prof. T. Ochiai for his helpful advice and encouragement.

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