111. On Regularity Properties for some Nonlinear Parabolic Equations*)

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The contents of this paper consist of some amelioration and supplement to the previous paper [4].

Let Ω be a not necessarily bounded domain in R^N, $N>2$, which is uniformly regular of class C^2 and locally regular of class C^4 in the sense of F. E. Browder [1]. The boundary of Ω is denoted by Γ. Let

$$a(u,v)=\int_\Omega\left(\sum_{i,j=1}^N a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+\sum_{i=1}^N b_i\frac{\partial u}{\partial x_i}v+cuv\right)dx$$

be a bilinear form defined in $H^1(\Omega)\times H^1(\Omega)$. The coefficients a_{ij}, b_i are bounded and continuous in $\bar{\Omega}$ together with first derivatives and c is bounded and measurable in Ω. The matrix $\{a_{ij}(x)\}$ is uniformly positive definite in Ω. It is assumed that $c\geqq 0$, $c-\sum_{i=1}^N \partial b_i/\partial x_i\geqq 0$ a.e. in Ω.

Let $j(x,r)$ be a function defined on $\Gamma\times R$ such that for each fixed $x\in\Gamma$ $j(x,r)$ is a proper convex lower semicontinuous function of r and $j(x,r)\geqq j(x,0)=0$. The subdifferential of j with respect to r is denoted by β. We assume that for each $t\in R$ and $\lambda>0$ $(1+\lambda\beta(x,\cdot))^{-1}(t)$ is a measurable function of x (cf. B. D. Calvert-C. P. Gupta [2]). For a function u defined on Γ $j(u)$ denotes the function $j(x,u(x))$, $x\in\Gamma$.

Set

$$\Gamma_1=\{x\in\Gamma:\beta(x,0)=R\},\qquad \Gamma_2=\Gamma\backslash\Gamma_1.$$

Γ_1 is the part of Γ where the boundary condition is of Dirichlet type. We assume that $\sum_{i=1}^N b_i\nu_i\geqq 0$ on Γ_2 where $\nu=(\nu_1,\cdots,\nu_N)$ is the outernormal vector to Γ. Set

$$V=\{u\in H^1(\Omega):u=0 \text{ on } \Gamma_1\}.$$

Let $\Psi(x)$ be a function belonging to $H^1(\Omega)\cap L^1(\Omega)$ such that $\Psi\leqq 0$ on Γ_1. We assume that

$$\{u\in V:u\geqq\Psi \text{ a.e.}, j(u|_\Gamma)\in L^1(\Gamma)\}$$

is not empty, or equivalently $j(\Psi^+|_\Gamma)\in L^1(\Gamma)$.

The norm of $L^2(\Omega)$ and $H^1(\Omega)$ are denoted by $|\ \ |$ and $\|\ \ \|$ respectively. The inner product of $L^2(\Omega)$ as well as the pairing between V and V^* are both denoted by $(\ ,\ )$. The norm of $L^p(\Omega)$ is denoted by $|\ \ |_p$.

The mapping A which is multivalued in general is defined as fol-

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lows: $u \in D(A)$ and $Au \ni f$ if $f \in L^2(\Omega)$, $\Psi \leqq u \in V$, $j(u|_\Gamma) \in L^1(\Gamma)$ and

$$a(u, v-u)+\int_\Gamma j(v|_\Gamma)d\Gamma-\int_\Gamma j(u|_\Gamma)d\Gamma \geqq (f, v-u)$$

for every v such that $\Psi \leqq v \in V$, $j(v|_\Gamma) \in L^1(\Gamma)$.

It is easily shown that A is maximal monotone in $L^2(\Omega)$, and $\overline{D(A)} = \{u \in L^2(\Omega) : u \geqq \Psi \text{ a.e.}\}$.

For $\Psi \leqq u_0 \in L^2(\Omega)$ and $f \in W^{1,1}(0, T; L^2(\Omega))$ set

$$S_f(t)u_0=\lim_{n\to\infty}\prod_{i=1}^{n}\left[1+\frac{t}{n}\left(A-f\left(\frac{i}{n}t\right)\right)\right]^{-1}u_0$$

(cf. M. G. Crandall-A. Pazy [3]).

Lemma 1. *If $f, \hat{f} \in L^2(\Omega)$, $f-\hat{f} \in L^p(\Omega)$, $1 \leqq p < \infty$, then for $\lambda > 0$ $(1+\lambda A)^{-1}f-(1+\lambda A)^{-1}\hat{f} \in L^p(\Omega)$ and*

$$|(1+\lambda A)^{-1}f-(1+\lambda A)^{-1}\hat{f}|_p \leqq |f-\hat{f}|_p.$$

From Lemma 1 the following proposition readily follows.

Proposition 1. *If $u_0, \hat{u}_0 \in L^2(\Omega)$, $u_0-\hat{u}_0 \in L^p(\Omega)$, $1 \leqq p < \infty$, then $S_f(t)u_0-S_f(t)\hat{u}_0 \in L^p(\Omega)$ and*

$$|S_f(t)u_0-S_f(t)\hat{u}_0|_p \leqq |u-\hat{u}_0|_p.$$

Lemma 2. *If $Au \ni f$, $A\hat{u} \ni \hat{f}$, $u-\hat{u} \in L^p(\Omega)$, $1<p<2$, then*

$$\begin{aligned}&(f-\hat{f}, |u-\hat{u}|^{p-2}(u-\hat{u}))+|u-\hat{u}|_p^p\\ &\quad\geqq\begin{cases}c_p|u-\hat{u}|_{pN/(N-2)}^p & \text{if } N>2\\ c_{p,q}|u-\hat{u}|_q^p \text{ for any } q\in[2,\infty) & \text{if } N=2.\end{cases}\end{aligned}$$

Using Lemma 2 and following the argument of L. Véron [5], pp. 175–176 we get

Proposition 2. *Suppose $\Psi \leqq u_0 \in L^2(\Omega)$, $\Psi \leqq \hat{u}_0 \in L^2(\Omega)$, $u_0-\hat{u}_0 \in L^p(\Omega)$, then*

$$\begin{aligned}&|S_f(t)u_0-S_f(t)\hat{u}_0|\\ &\quad\leqq\begin{cases}c_p(1+t^{-N(p^{-1}-2^{-1})/2})|u_0-\hat{u}_0|_p & \text{if } N>2\\ c_{p,\sigma}(1+t^{-\sigma})|u_0-\hat{u}_0|_p \text{ for any } \sigma>p^{-1}-2^{-1} & \text{if } N=2.\end{cases}\end{aligned}$$

From Propositions 1 and 2 we get the following result.

Theorem 1. *For $f \in W^{1,1}(0, T; L^2(\Omega))$ the mapping $S_f(t)$ can be extended to a mapping from $\{u \in L^p(\Omega) : u \geqq \Psi$ a.e.$\}$ to $L^2(\Omega)$ for any $1 \leqq p < 2$. For any u_0 such that $\Psi \leqq u_0 \in L^p(\Omega)$, $1 \leqq p < 2$, $S_f(t)u_0 \to u_0$ in $L^p(\Omega_R)$ as $t \to 0$ for any $R>0$ where $\Omega_R=\Omega\cap\{x : |x|<R\}$.*

Let $\tilde{A}$ be the operator defined as A with j replaced by the function $\tilde{j}$ such that $\tilde{j}(x, \cdot)=j(x, \cdot)=$ the indicator function of $\{0\}$ for $x \in \Gamma_1$ and $\tilde{j}(x, \cdot)\equiv 0$ for $x \in \Gamma_2$. Namely the boundary condition on Γ_2 is replaced by that of Neumann type by this replacement.

Let L and $\mathcal{L}$ be the linear operators on V to V^* and $H^1(\Omega)$ to V^* defined by

$$(Lu, v)=a(u, v),\ u, v \in V,$$
$$(\mathcal{L}u, v)=a(u, v),\ u \in H^1(\Omega),\ v \in V$$

respectively. Let w be the solution of the equation

$$w'+\tilde{A}w \ni f^+,\ w(0)=u_0^+,$$

and v be the solution of the linear equation in V^*

$$v'+Lv=\mathcal{L}\Psi+f^+,\ v(0)=u_0^+.$$

Then it is shown that

$$\Psi\leqq S_f(\cdot)u_0\leqq(w-v)^+ +v.$$

Hence we get

Theorem 2. *For $\Psi\leqq u_0\in L^p(\Omega)$, $1\leqq p\leqq 2$, we have for $0<t\leqq T$*

$$|S_f(t)u_0|\leqq C(t^{N(2^{-1}-p^{-1})/2}|u_0^+|_p+t^{1/2}\|\Psi\|)+|\Psi|+\int_0^t|f^+(s)|\,ds.$$

The right derivative of $S_f(t)u_0$ exists in $(0,T]$. Arguing as in [4] we get

Theorem 3. *If in addition to the assumption of Theorem 2 f belongs to $W^{1,1}(0,T;L^r(\Omega))$, $r\geqq 2$, then the right derivative $D^+S_f(t)u_0$ which exists in the strong topology of $L^2(\Omega)$ belongs to $L^r(\Omega)$, and*

$$\begin{aligned}|D^+S_f(t)u_0|_r\leqq C\Big\{&t^{-\gamma-1}|u_0^+|_p+t^{-\alpha-1}(|\Psi|+t^{1/2}\|\Psi\|+|v|+t|A^0v|)\\&+t^{-\alpha-1}\Big(\int_0^t|f(s)|\,ds+\int_0^t s|f'(s)|\,ds\Big)\\&+\int_0^t|f'(s)|_r\,ds\Big\}\end{aligned}$$

where $\gamma=N(p^{-1}-r^{-1})/2$, $\alpha=N(2^{-1}-r^{-1})/2$, v is an arbitrary element of $D(A)$ and A^0 is the minimal cross-section of A.

In what follows we assume that either Ω is bounded or there exists a function $\tilde{\Psi}\in L^1(\Omega)$ such that $a(\Psi,v)\leqq(\tilde{\Psi},v)$ for any v satisfying $0\leqq v\in V\cap L^\infty(\Omega)$. The latter condition is satisfied if $\Psi\in W^{2,1}(\Omega)$, $\mathcal{A}\Psi\in L^1(\Omega)$, $\partial\Psi/\partial n\leqq 0$ on Γ_2, where $\mathcal{A}$ is the linear differential operator associated with the bilinear form $a(u,v)$ and $\partial/\partial n$ is the conormal derivative with respect to $\mathcal{A}$.

Theorem 4. *Under the assumptions stated above the mapping A_p defined by*

$G(A_p)=$the closure of $G(A)\cap(L^p(\Omega)\times L^p(\Omega))$ in $L^p(\Omega)\times L^p(\Omega)$

where $G(A)$ denotes the graph of A is m-accretive in $L^p(\Omega)$ for $1\leqq p<2$, and

$$\overline{D(A_p)}=\{u\in L^p(\Omega):u\geqq\Psi\ \text{a.e.}\}.$$

Under the assumptions of Theorem 4 if $\Psi\leqq u_0\in L^p(\Omega)$ and $f\in W^{1,1}(0,T;L^p(\Omega))$, then $S_f(t)u_0\in L^p(\Omega)$ and $S_f(t)u_0\to u_0$ in $L^p(\Omega)$ as $t\to 0$.

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