72. On Asymptotic Equivalence of Bounded Solutions of Two Integro-Differential Equations

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Abstract. In this note we consider some problems concerning the asymptotic equivalence of bounded solutions of integro-differential equations.

Consider the perturbed system of integro-differential equations

$$\text{(P)} \qquad x'(t)=A(t)x(t)+\int_{t_0}^{t} B(t,s)x(s)ds+f(x)(t), \qquad t\geqq t_0,$$

where A, B are given $n\times n$ matrices and the perturbation n vector $f(x)(\,.\,)$ is an operator mapping the set of functions defined for $t\geqq t_0$ into itself; for example, typical perturbations are of the form

$$f(x)(t)=f(t,x(t)) \quad \text{or} \quad \int_{t_0}^{t} K(t,s,x(s))ds$$

$$\text{or} \quad x(t)\int_{t_0}^{t} K(t,s,x(s))ds.$$

We are interested in comparing the bounded solutions of (P) with those of the related unperturbed linear system

$$\text{(L)} \qquad y'(t)=A(t)y(t)+\int_{t_0}^{t} B(t,s)y(s)ds, \qquad t\geqq t_0.$$

In particular, we will determine conditions on A, B and f so that each bounded solution y of (L) corresponds to a bounded solution x of (P), in such a way that their difference $y-x$ tends to zero asymptotically, and conversely, each bounded solution x of (P) corresponds to a bounded solution y of (L) such that their difference $x-y$ tends again to zero asymptotically. In other words, the systems (P) and (L) should be asymptotically equivalent.

J. A. Nohel ([7], [8]) has established the asymptotic equivalence of (P) and (L) in the case that the linear system (L) is asymptotically stable. Our aim is to cover the cases that the linear system (L) is conditionally asymptotically stable, conditionally uniformly asymptotically stable and conditionally uniformly stable.

The fundamental solution matrix (or resolvent kernel) of (L) is the solution $Y(t,s)$ of the matrix equation

$$\frac{\partial}{\partial t}Y(t,s)=A(t)Y(t,s)+\int_{s}^{t} B(t,r)Y(r,s)dr, \qquad t\geqq s\geqq t_0,$$

$$Y(s,s)=I,$$

λ, $0<\lambda<\infty$,

$$\lim_{t\to\infty}\sup_{s\geqq t}\omega(s,\lambda)=0, \tag{1}$$

then, corresponding to each bounded solution y of (L), *there exists a bounded solution x of* (P) *such that*

$$\lim_{t\to\infty}|x(t)-y(t)|=0. \tag{2}$$

Conversely, to each bounded solution x of (P) *there corresponds a bounded solution y of* (L) *such that* (2) *holds.*

Theorem 2. *Suppose that H_1^q and H_3 hold for $1<q<\infty$. If for every λ, $0<\lambda<\infty$, and p, $1<p<\infty$, $p^{-1}+q^{-1}=1$,*

$$\int_0^\infty \omega^p(s,\lambda)ds<\infty, \tag{3}$$

then the conclusions of Theorem 1 *remain true.*

The above theorems in the case $p\equiv 0$ (i.e. (L) asymptotically stable) are reduced to the results of Nohel ([7], [8]). In the differential equations case ($B\equiv 0$) they are reduced to the results of Coppel [2] and Hallam [5]. Note that in the latter case, it turns out that $P(t)=-P_2Y^{-1}(t)$, where $Y(t)$ is a fundamental solution matrix, and so $W(t,s)=Y(t)P_2Y^{-1}(s)$ and $V(t,s)=Y(t)P_1Y^{-1}(s)$, where P_1, P_2 are complementary projections.

The conditions on ω in Theorem 2 can be slightly extended in the cost of slightly restricting the hypothesis H_1^q. Thus the next theorem generalizes to integro-differential equations a result of Lovelady [6] for differential equations.

Theorem 3. *Suppose that H_3 holds and that there exist constants K, q, with $K>0$ and $1<q<\infty$, such that for each $t\geqq t_0$*

$$\left\{\sum_{k=t_0}^{t}\int_k^{k+1}|V(t,s)|\,ds\right\}^{1/q}+\left\{\sum_{k=t}^{\infty}\int_k^{k+1}|W(t,s)|\,ds\right\}^{1/q}\leqq K. \tag{4}$$

If for every λ, $0<\lambda<\infty$, and p, $1<p<\infty$, $p^{-1}+q^{-1}=1$,

$$\lim_{t\to\infty}\int_t^{t+1}\omega^p(s,\lambda)ds=0, \tag{5}$$

then the conclusions of Theorem 1 *remain true.*

The next theorem establishes the asymptotic equivalence of (P) and (L) under conditions which imply that (L) is conditionally uniformly asymptotically stable.

Theorem 4. *Suppose that H_1^∞ and H_3 hold. If for every λ, $0<\lambda<\infty$, either* (1) *or* (3) *is valid, the latter for p such that $1<p<\infty$, then the conclusions of Theorem* 1 *remain true.*

When the linear system (L) is conditionally uniformly stable, the following theorem establishes the asymptotic equivalence of (P) and (L) generalizing the differential equations result of Brauer and Wong [1].

Theorem 5. *Suppose that H_2^∞ and H_3 hold. If for every λ, $0<\lambda<\infty$,* where I is the $n\times n$ identity matrix. If $s>t\geqq t_0$, we define $Y(t,s)\equiv 0$. In what follows, we will assume that $A(t)$ and $B(t,s)$ are locally integrable for $t\geqq t_0$ and $t\geqq s\geqq t_0$ respectively. Then (cf. [8]), $Y(t,s)$ exists, it is continuous and, for locally integrable perturbations $f(x)(.)$, (P) is equivalent to the following Volterra integral equation

$$x(t)=Y(t,t_0)x(t_0)+\int_{t_0}^{t} Y(t,s)f(x)(s)ds, \qquad t\geqq t_0.$$

We will assume that there exists an $n\times n$ matrix $P(t)$, locally integrable for $t\geqq t_0$, in terms of which we define the following matrices (in the notation of [3])

$$\begin{aligned} V(t,s)&=Y(t,s)-Y(t,t_0)P(s), && t_0\leqq s\leqq t,\\ W(t,s)&=-Y(t,t_0)P(s), && t_0\leqq t\leqq s. \end{aligned}$$

Concerning the linear system (L) we make the following hypotheses

H_1^q: there exist constants K, q, with $K>0$ and $1\leqq q<\infty$, such that for each $t\geqq t_0$

$$\left\{\int_{t_0}^{t} |V(t,s)|^q\, ds\right\}^{1/q}+\left\{\int_{t}^{\infty} |W(t,s)|^q\, ds\right\}^{1/q}\leqq K;$$

H_1^∞: there exist constants K_1, K_2, a_1, a_2, all positive, such that for each $t\geqq t_0$

$$\begin{aligned} |V(t,s)|&\leqq K_1e^{-a_1(t-s)}, && t_0\leqq s\leqq t,\\ |W(t,s)|&\leqq K_2e^{-a_2(s-t)}, && t_0\leqq t\leqq s; \end{aligned}$$

H_2^∞: there exists a constant $K>0$ such that for each $t\geqq t_0$

$$\begin{aligned} |V(t,s)|&\leqq K, && t_0\leqq s\leqq t,\\ |W(t,s)|&\leqq K, && t_0\leqq t\leqq s; \end{aligned}$$

H_3: for each fixed $T\geqq t_0$

$$\lim_{t\to\infty}\int_{t_0}^{T} |V(t,s)|\, ds=0.$$

J. M. Cushing ([3], [4]) has shown that the above hypotheses are necessary and sufficient conditions for the admissibility of certain function spaces and for conditional stability of (L). Moreover, he has shown that conditional stability is preserved for (P) under appropriate perturbations $f(x)(.)$.

Let C denote the Banach space of continuous and bounded vector functions $u(t)$ for $t\geqq t_0$. The norm of $u\in C$ is $\|u\|=\sup_{t\geqq t_0}|u(t)|$.

As for the perturbation $f(x)(.)$, we assume that $f: C\to C$ is continuous and such that for any $x\in C$ and $t\geqq t_0$

$$|f(x)(t)|\leqq \omega(t,\|x\|),$$

where $\omega(t,r)$ is a given nonnegative function which is continuous in $t\geqq 0$ for each fixed $r\geqq 0$ and nondecreasing in $r\geqq 0$ for each fixed $t\geqq 0$.

Now we are in the position to establish the asymptotic equivalence of (P) and (L) under conditions which imply that (L) is conditionally asymptotically stable.

Theorem 1. *Suppose that H_1^q and H_3 hold for $q=1$. If for every*

$$\int_0^\infty \omega(s,\lambda)\,ds<\infty, \tag{6}$$

then the conclusions of Theorem 1 *remain true.*

The proofs of all the above theorems are similar to those of the differential equations cases (cf. [1], [2], [5], [6]). Here we will only indicate the essential steps of proof.

On an appropriate closed ball S of C we define, for given $y\in C$ solution of (L), the mapping

$$Tx(t)=y(t)+\int_{t_0}^t V(t,s)f(x)(s)ds+\int_t^\infty W(t,s)f(x)(s)ds.$$

Using the hypotheses on V, W and ω, we obtain (through Hölder's inequality) that T is a continuous mapping of S into itself. Clearly TS is uniformly bounded. Since $z=Tx$ solves the integral equation

$$z(t)=Y(t,t_0)\left\{y(t_0)-\int_{t_0}^\infty P(s)f(x)(s)ds\right\}+\int_{t_0}^t Y(t,s)f(x)(s)ds,$$

it follows that z solves the nonhomogeneous linear system

$$z'(t)=A(t)z(t)+\int_{t_0}^t B(t,s)z(s)ds+f(x)(t),$$

i.e. TS is equicontinuous. Hence, Schauder's Fixed Point theorem implies the existence of a solution $x\in C$ of (P), which is easily seen to verify (2). The converse is quite simple.

References

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