63. A Generalized Poincaré Series Associated to a Hecke Algebra of a Finite or p-Adic Chevalley Group*)

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Introduction. Let (W, S) be a Coxeter system ([1]) with finite generator system S. The Poincaré series of W is by definition the formal power series $\sum_{w\in W} t^{l(w)}$, in which t is a variable and $l(w)$ is the length of w with respect to the generator system S of W. This series has arisen in works of many authors (see the references of [4]). Our main purpose is to investigate the properties of the formal power series of matrix coefficients $L(t, R)=L(t, q, W, R)$ defined by (♯) in § 1 for a representation R of the Hecke algebra H_q $(q>0)$ (see § 1 for the definition of H_q). (Note that if $q=1$ and R is trivial, $L(t, R)$ is just the Poincaré series (W, S).) In particular we show that $L(t, R)$ is similar, in property, to the congruence zeta function of an algebraic variety. See 1)–3) below. The original motivation of this work was to associate a kind of L-function to an irreducible representation of the Hecke algebra H_q (hence, to an irreducible constituent of the natural representation of G on the space of functions on G/B, where G is a finite (resp. p-adic) Chevalley group and B is a Borel (resp. Iwahori) subgroup of G). The main results of this paper are:

1) Components of $L(t, R)$ are rational functions (Theorem 1),

2) if W is finite,

i) the function $L(t, R)$ satisfies a functional equation (Theorem 2. (1)),

ii) the absolute values of the zeros of det $L(t, R)$ are of the forms q^{-a} for some rational numbers $0\leq a\leq 1$ (Theorem 2. (2)),

iii) the zeros on the boundary of 'the critical strip' can be described explicitly in terms of vertices of W-graph ([3]), if R has a W-graph (Theorem 3).

(The author can prove that any finite dimensional representation of a finite irreducible Coxeter group has a W-graph with the possible exception of the Coxeter group of type H_4. The details will be published elsewhere.)

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3) Let W be a Weyl group of type A_n and W_a the corresponding affine Weyl group. We construct an algebra homomorphism A of $H_q(W_a)$ onto $H_q(W)$ (Theorem 4). We show that $L(t, W_a, R\circ A)$ also has a functional equation and its zeros are of the forms q^{-a} ($0\leq a\leq 1$, $a\in \boldsymbol{Q}$) (Theorem 5). A relation between $L(t, W_a, R\circ A)$ and $L(t, W, R)$ is also given in Theorem 5.

All proofs are omitted and will be published elsewhere.

1. Let (W, S) be a Coxeter system with finite generator system S ([1]). For an element w in W, $l(w)$ denotes the length of w with respect to S. For a positive real number q, the Hecke algebra $H_q=H_q(W)$ is by definition the associative $\boldsymbol{C}$-algebra with a basis $\{e_w\}_{w\in W}$, and relations

$$e_s e_w=\begin{cases} e_{sw}, & \text{if } l(sw)>l(w),\\ (q-1)e_w+qe_{sw}, & \text{if } l(sw)<l(w),\end{cases}$$

[1, p. 55, Ex. 23]. This algebra H_q has an involutory automorphism defined by

$$\hat{e}_w=(-q)^{l(w)}(e_{w^{-1}})^{-1},$$

(see [2]). For a finite dimensional representation R of H_q, we set

$$(\sharp)\qquad L(t, q, W, R)=\textstyle\sum_{w\in W} R(e_w)t^{l(w)}.$$

Sometimes we write $L(t, R)$, $L(t, q, R)$ or $L(t, W, R)$ for $L(t, q, W, R)$.

Theorem 1. *The matrix components of $L(t, R)$ are rational functions in t.*

2. Let R be a finite dimensional representation of H_q, $\hat{R}$ the representation defined by $\hat{R}(e_w)=R(\hat{e}_w)$ for every w in W, and if W is finite, N the length of the unique longest element w_0 of W.

Theorem 2. *Let W be a finite Coxeter group.*

(1) *We have the equality*

$$L(t, \hat{R})=R(e_{w_0})^{-1}(-qt)^N\cdot L((-qt)^{-1}, R).$$

(2) *The absolute values of the zeros of* $\det L(t, R)$ *are of the forms $q^{-i/m}$ with some integers i and m such that $1\leq m\leq 2N$ and $0\leq i/m\leq 1$.*

Let $\Gamma=(X, Y, I, \mu)$ be a finite W-graph ([3]), where X is the set of vertices and Y the set of edges and R the corresponding representation of H_q. Put $L(t, \Gamma)=L(t, R)$. Linear characters sgn and ind are defined by $\operatorname{sgn} e_w=(-1)^{l(w)}$ and $\operatorname{ind} e_w=q^{l(w)}$. Operators $L_0(t, \Gamma)$, $L_1(t, \Gamma)$ and $L^0(t, \Gamma)$ on the space $\sum_{x\in X} \boldsymbol{C}x$ are defined by

$$L_0(t, \Gamma)x=L(t, W_{I_x}, \operatorname{sgn})x,$$
$$L_1(t, \Gamma)x=L(t, W_{S-I_x}, \operatorname{ind})x,$$
$$L^0(t, \Gamma)=L_1(t, \Gamma)^{-1}L(t, \Gamma)L_0(t, \Gamma)^{-1}.$$

Theorem 3. *Let W be a finite Coxeter group.*

(1) *The operator $L^0(t, \Gamma)$ is represented by a matrix, with respect to the basis $\{x\}_{x\in X}$, whose components are polynomials in $q^{1/2}$ and t.*

(2) *The absolute values of zeros of* $\det L^0(t, \Gamma)$ *are of the forms*

$q^{-i/m}$ *with some integers* i *and* m *such that* $1 \leq m \leq 2N$ *and* $0 < i/m < 1$, i.e., *all the zeros of* $\det L(t, \Gamma)$ *on the boundary of 'the critical strip' come from the factors* $\det L_0(t, \Gamma)$ *and* $\det L_1(t, \Gamma)$.

Example. If W is of type A_3 and Γ is ①—②—③ (see [3] for this expression), then

$$L_1(t, \Gamma) = \mathrm{diag}\,((1+qt)(1+qt+q^2t^2),\ (1+qt)^2,\ (1+qt)(1+qt+q^2t^2)),$$

$$L_0(t, \Gamma) = \mathrm{diag}\,(1-t, 1-t, 1-t),$$

$$L^0(t, \Gamma) = \begin{bmatrix} 1 & q^{1/2}t & qt^2 \\ q^{1/2}t(1+qt) & 1+q^2t^3 & q^{1/2}t(1+qt) \\ qt^2 & q^{1/2}t & 1 \end{bmatrix},$$

$$\det L^0(t, \Gamma) = (1-qt^2)^2(1-q^2t^3).$$

More generally, if W is of type A_n and Γ is ①—②—$\cdots$—ⓝ, then

$$\det L^0(t, \Gamma) = \prod_{i=1}^{n-1} (qt^2; qt)_i,$$

where

$$(x; y)_i = (1-x)(1-xy)\cdots(1-xy^{i-1}).$$

3. Let W_a be the affine Weyl group of type A_n and $S_a = \{s_0, s_1, \cdots, s_n\}$ the set of canonical generators which is numbered in a circular order. Let $e_i = e_{s_i}$, $S = \{s_1, \cdots, s_n\}$ and W the group generated by S.

Theorem 4. *There is a homomorphism* A *of* $H_q(W_a)$ *onto* $H_q(W)$ *such that*

$$Ae_0 = e_1e_2\cdots e_{n-1}e_ne_{n-1}^{-1}\cdots e_2^{-1}e_1^{-1},$$

$$Ae_i = e_i \qquad (1 \leq i \leq n).$$

Remark. The above homomorphism A specializes to the natural homomorphism $W_a \to W$ when q specializes to 1. In general, let W_a (resp. W) be the affine Weyl group (resp. Weyl group) of an irreducible root system Σ. Then no homomorphism $H_q(W_a) \to H_q(W)$ specializes to the natural homomorphism $W_a \to W$ when q specializes to 1, unless Σ is of type A_n.

Theorem 5. (1) *Let* R *be a finite dimensional representation of* $H_q(W)$. *Then*

$$\det L(t, W, R)^{\deg R} / \det L(t, W_a, R \circ A)$$

is a polynomial in t.

(2) *We have the equality*

$$\det L(t, W_a, \hat{R} \circ A) = \pm q^a t^b \det L((-qt)^{-1}, W_a, R \circ A)$$

with some integers a *and* b.

(3) *The absolute values of the poles of* $\det L(t, W_a, R \circ A)$ *are of the forms* $q^{-i/m}$ *with some integers* i *and* m *such that* $1 \leq m \leq n^2(n+1)$ *and* $0 \leq i/m \leq 1$.

Example. Let W be the Weyl group of type A_2 and R the irreducible representation of degree 2. Then

$$\det L(t, W, R) = (1-t)^2(1-qt^2)(1+qt)^2,$$

$$\det L(t, W_a, R \circ A) = (1-t)^2(1-qt^2)^2(1+qt)^2 \cdot \{(1+t+t^2)(1+qt^2+q^2t^4)(1-qt+q^2t^2)\}^{-1}.$$

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