50. *The Ring and Module Forms of the Brauer Correspondence*

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The Brauer correspondence has a ring theoretic definition by means of central characters and the Brauer map (see [3], III. 9). There is a module theoretic definition in terms of restriction of modules (see [2, 5, 1]). The natural question arises as to how the definitions differ. In [1] Alperin showed by non-elementary techniques that the two definitions coincide in the case of primary interest, i.e., for blocks of a subgroup H of G having defect group D with $C_G(D) \leq H$. In [5] Okuyama used elementary techniques to obtain a broadening of Alperin's result. He showed the two correspondences are defined and agree for a block b of a subgroup H of G, if b has multiplicity one as a direct summand of the group algebra FG.

The first section refines the techniques of the Okuyama article. We obtain a strikingly clear contrast between the two definitions in Remark 1.6. One immediate consequence is that whenever both forms of the Brauer correspondence are defined, they coincide (see Theorem 1.7). The remaining two sections address the natural question as to possible differences between the domain of definition of the two forms of the Brauer correspondence. We show that there are differences suggesting the potential usefulness of each definition in a general setting.

1. A general comparison. Fix a finite group G and a subgroup H. Let F be a field of characteristic p. By "block of FH" we mean one of the ideals in the direct sum decomposition of FH into indecomposable two sided ideals; or analogously, an indecomposable $F(H \times H)$-submodule of FH that is a direct summand (see [2] for a complete treatment of the module view). We view FH as a subset of FG.

Definition 1.1. For any block b of FH, define

$$\theta_b : \mathrm{End}_{F(G\times G)}(FG) \to \mathrm{End}_{F(H\times H)}(b)$$

by $\theta_b(f) = \pi_b f \rho_b$ where $\pi_b : FG \to b$ is the $F(H\times H)$-module projection and $\rho_b : b \to FG$ is the $F(H\times H)$-module injection. Also let

$$\bar{\theta}_b : \mathrm{End}_{F(G\times G)}(FG) \to \mathrm{End}_{F(H\times H)}(b)/J(\mathrm{End}_{F(H\times H)}(b))$$

be θ_b composed with the canonical map.

For b a block of FH, b^G indicates its ring theoretic Brauer correspondent. In general $\bar{\theta}_b$ is only a F-vector space homomorphism.

Proposition 1.2 (Okuyama). *For b a block of FH, the following are equivalent.*

(a) *b^G is defined.*

(b) *$\bar{\theta}_b$ is a ring homomorphism.*

Proof. Since the endomorphism rings of FG and b in the difinition of θ_b are isomorphic to the centers $Z(FG)$ and $Z(b)$, the map $\bar{\theta}_b$ is the "same" as the Brauer map $Z(FG)\to Z(b)$ composed with a central linear character of b. (See [5, Prop. 1] for details. Note the assumption there that F is a splitting field. Using the ring theoretic approach to the Brauer correspondence developed in [4, § 7] the assumption can be seen as unnecessary.)

To characterize the module form of the Brauer correspondence, Definition 1.1 must be broadened.

Definition 1.3. For U any internal $F(H\times H)$-module direct summand of FG, define

$$\theta_U : \mathrm{End}_{F(G\times G)}(FG)\to \mathrm{End}_{F(H\times H)}(U)$$

by $\theta_U(f)=\pi_U f\rho_U$ where $\pi_U : FG\to U$ is the $F(H\times H)$-module projection and $\rho_U : U\to FG$ is the $F(H\times H)$-module injection. Also let

$$\bar{\theta}_U : \mathrm{End}_{F(G\times G)}(FG)\to \mathrm{End}_{F(H\times H)}(U)/J(\mathrm{End}_{F(H\times H)}(U))$$

be θ_U composed with the canonical map.

Of course

$$\mathrm{End}_{F(G\times G)}(FG)\cong \oplus \mathrm{End}_{F(G\times G)}(B)$$

where B runs over all blocks of FG. We view $\mathrm{End}_{F(G\times G)}(B)$ as a subset of $\mathrm{End}_{F(G\times G)}(FG)$ via this isomorphism. The following lemma gives the key property.

Lemma 1.4. *Let B be a block of FG and U be an indecomposable $F(H\times H)$-module. The following are equivalent.*

(a) *U is isomorphic to a direct summand of $B_{H\times H}$.*

(b) *$\bar{\theta}_{U'}(\mathrm{End}_{F(G\times G)}(B))\neq 0$ for some U' an internal direct summand of $(FG)_{H\times H}$ with $U\cong U'$.*

Proof. Condition (a) implies there is some U' and internal direct summand of $B_{H\times H}$ with $U'\cong U$. Clearly $\theta_{U'}(\mathrm{End}_{F(G\times G)}(B))$ contains the scalar multiplication maps on U'. Thus condition (b) holds. Conversely, if we let $f\in \mathrm{End}_{F(G\times G)}(B)$ be a map with $\bar{\theta}_{U'}(f)\neq 0$, then, since $U'\cong U$ is indecomposable, $\theta_{U'}(f)=\pi_{U'}f\rho_{U'}$ is an automorphism of U'. But then we have a forwards map $f\rho_{U'} : U'\to B$ and a backwards map $\pi_{U'} : B\to U'$ that compose to an automorphism of U'. This gives condition (a).

For a block b of FH, $m^G b$ indicates the module theoretic Brauer correspondent of b.

Proposition 1.5. *Let b be a block of FH and B be a block of FG. The following are equivalent.*

(a) $m^G b=B$.

(b) $\bar{\theta}_{U'}(\mathrm{End}_{F(G\times G)}(B'))\begin{cases}=0 & \text{if } B'\neq B\\ \neq 0 & \text{if } B'=B\end{cases}$

for all U' internal direct summands of $(FG)_{H\times H}$ with $U'\cong b$.

Proof. Immediate from Lemma 1.4 and the definition of $m^G b$.

Remark 1.6. It is now easy to summarize the difference between the two forms of the Brauer correspondence. First note that in Proposition 1.5 that if $\bar{\theta}_{U'}$ is a ring homomorphism, then condition (b) holds for that particular U'. Thus if we compare condition (b) in Proposition 1.2 and Lemma 1.4, we see that for b^G to be defined we need a stronger condition on $\bar{\theta}_{U'}$ for any other direct sum embeddings of b in $(FG)_{H\times H}$. For $m^G b$ to be defined a weaker condition must hold, but uniformly for all direct sum copies of b in $(FG)_{H\times H}$.

We easily obtain the main theorem.

Theorem 1.7. *Let b be a block of FH. Whenever b^G and $m^G b$ are both defined, they are equal.*

Proof. In both cases the Brauer correspondent is the block not in the kernel of $\bar{\theta}_b$.

Thus the only way in which the module and ring definitions of the Brauer correspondence can differ is in their domain of definition. As we shall see in the next two sections, this is very much the case.

2. The Brauer correspondences and normal subgroups. We shall examine the two forms of the Brauer correspondence between the finite group G and a normal subgroup K. In this setting one also has the "covering" relationship between blocks. For the notion of "covering", the module and ring approaches coincide completely. (See [2, § 4] for a module treatment and [4, § 6] for a ring treatment.)

Lemma 2.1. *For any block b of FK, $m^G b$ is defined if and only if b is covered by only one block of FG.*

This lemma is immediate since a block B of FG covers b if and only if b is a direct summand of $B_{K\times K}$. Essentially, the module form of the Brauer correspondence is the extension of the natural idea of covering in the case of a normal subgroup.

Recall that a block B of FG with central character λ is regular with respect to K if and only if $\lambda(C)=0$ for all G-class sums C not in K. From [3, V. 3.6] we get the characterization of the ring form of the Brauer correspondence for normal subgroups.

Lemma 2.2. *For any block b of FK, b^G is defined if and only if there is a regular block of FG covering b.*

For a block b of FK, being covered by only one block and being covered by a regular block are closely allied, but different ideas. We get the following examples of the differences in the domain of difinition of the two correspondences.

Example 1. Let $G=SL(2, 5)$, $K=Z(G)$ and $p=2$. Then FG has two blocks B_0 and B_1. Both blocks cover the only block b_0 of FK. Hence by Lemma 2.1, $m^G b_0$ is not defined. However, it is easy to see that the principal block B_0 of FG is regular with respect to K. This is because the only conjugacy classes in G with a p' number of elements are in K. Hence b_0^G is defined by Lemma 2.2.

Example 2. Let G be such that FG has only one block B_0. Let K be a normal subgroup not containing some class of maximal defect. (The easiest examples of this are G a p-group, K a normal subgroup not containing the center, or G the symmetric group on three letters, $K=1$, $p=3$.) Then $m^G b$ is defined for all blocks b of FK by Lemma 2.1. However the central character for B_0 will not be zero on the conjugacy class of maximal defect that is not in K. Hence B_0 is not regular with respect to K. By Lemma 2.2, b^G is not defined for all blocks b of FK.

3. The Brauer correspondences and arbitrary subgroups. Now we consider a finite group G and an arbitrary subgroup H. Again we shall compare results about the two correspondences. For the module form, the following is obvious.

Lemma 3.1. *If FG has only one block, then $m^G b$ is defined for all blocks b of FH.*

A much deeper result about the ring form is the following

Lemma 3.2. *If B is a block of FG and b is a block of FH, then $b^G=B$ implies* $\exp(Z(D(B)))\leq\exp(Z(D(b)))$.

See [3, V. 1.6] for this result.

If one takes G such that FG has only one block B_0, then we have $m^G b$ defined for all blocks b of any subgroup H. However from Lemma 3.2, if the exponent of the center of the Sylow p-subgroup of H is less than the exponent of the center of the Sylow p-subgroup of G, then b^G is never defined.

The contrast developed above makes the different potential uses of the two correspondences clear. As Lemma 3.2 indicates, $b^G=B$ is a very delicate condition which is closely linked to the group structure of G. The condition $m^G b=B$ is much cruder and more closely tied to the ring structure of FG.

References

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