18. *Block Intersection Numbers of Block Designs*

By Mitsuo YOSHIZAWA
Gakushuin University

(Communicated by Kunihiko KODAIRA, M. J. A., Feb. 12, 1980)

§ 1. **Introduction.** In this note, we shall assume throughout that the block designs are nontrivial. For a t-(v,k,λ) design $\boldsymbol{D}$ we use λ_i $(0 \leqslant i \leqslant t)$ to represent the number of blocks which contain the given i points of $\boldsymbol{D}$. A t-(v,k,λ) design $\boldsymbol{D}$ is called block-schematic if the blocks of $\boldsymbol{D}$ form an association scheme with the relations determined by size of intersection (cf. [3]). For a block B of a t-(v,k,λ) design $\boldsymbol{D}$ we use $x_i(B)$ $(0 \leqslant i \leqslant k)$ to denote the number of blocks each of which has exactly i points in common with B. If $x_i(B)$ is uniquely determined by the choice of a block B for each i $(i=0,\cdots,k)$, we shall say that $\boldsymbol{D}$ is block-regular, and we shall write x_i instead of $x_i(B)$. We remark that if a t-(v,k,λ) design $\boldsymbol{D}$ is block-schematic, then $\boldsymbol{D}$ is block-regular. In this note, we shall give the following two theorems. The detailed proofs will be given elsewhere.

Theorem 1. *For each $n \geqslant 1$ and $\lambda \geqslant 1$,*

(a) *there exist at most finitely many block-schematic t-(v,k,λ) designs with $k-t=n$ and $t \geqslant 3$, and*

(b) *if also $\lambda \geqslant 2$, there exist at most finitely many block-schematic t-(v,k,λ) designs with $k-t=n$ and $t \geqslant 2$.*

Theorem 2. *Let c be a real number with $c>2$. Then for each $n \geqslant 1$ and $l \geqslant 0$, there exist at most finitely many block-regular t-(v,k,λ) designs with $k-t=n$, $v \geqslant ct$ and $x_i \leqslant l$ for some i $(0 \leqslant i \leqslant t-1)$.*

Remark. Since there exist infinitely many 2-$(v,3,1)$ designs, and since every 2-$(v,k,1)$ design is block-schematic (cf. [2]), Theorem 1 does not hold for $\lambda=1$ and $t=2$.

§ 2. **Outline of the proof of Theorem 1.** Lemma 1. *Let $\boldsymbol{D}$ be a block-regular t-(v,k,λ) design. Then the following equation holds for $i=0,\cdots,k-1$.*

$$x_i=\sum_{j=i}^{t-1}\binom{j}{i}(\lambda_j-1)\binom{k}{j}(-1)^{i+j}+\sum_{j=t}^{k-1}\binom{j}{i}w_j(-1)^{i+j},$$

where $x_j \leqslant w_j \leqslant (\lambda-1)\binom{k}{j}$ $(t \leqslant j \leqslant k-1)$.

Lemma 2. *Let $\boldsymbol{D}$ be a t-(v,k,λ) design with $t,\lambda \geqslant 2$. If $v \geqslant k^3$, then there exist three blocks B_1, B_2, B_3 of $\boldsymbol{D}$ such that $|B_1 \cap B_2|=t-1$, $|B_2 \cap B_3| \geqslant t$ and $|B_1 \cap B_3|=t-2$.*

By making use of Lemmas 1, 2 and the idea of Atsumi [1], we get the following

Proposition. *Let D be a block-schematic t-(v, k, λ) design with $t, \lambda \geqslant 2$. Then $v<\lambda k^3\binom{k}{[k/2]}^2$ holds.*

Lemma 3. *For each $n \geqslant 1$, there is a positive integer $N_1(n)$ satisfying the following: If D is a t-(v, k, λ) design with $k-t=n$ and $t \geqslant N_1(n)$, then there exist two blocks B_1 and B_2 of D such that $|B_1 \cap B_2|=t-1$.*

Lemma 4. *For each $n \geqslant 1$, there is a positive integer $N_2(n)$ satisfying the following: If D is a t-(v, k, λ) design with $k-t=n$ and $t \geqslant N_2(n)$, then there exist three blocks B_1, B_2, B_3 of D such that $|B_1 \cap B_2|=t-1$, $|B_2 \cap B_3|=t-1$ and $|B_1 \cap B_3|=t-n-2$.*

By making use of Lemma 1, Proposition, Lemma 4 and [1, Theorem], we prove Theorem 1.

§3. Outline of the proof of Theorem 2. Lemma 5. *Let D be a block-regular t-(v, k, λ) design. Then the following equation holds for $i=0, \cdots, t-1$.*

$$x_i=\frac{\lambda\binom{k}{i}}{\binom{v-t}{k-t}}\left\{\binom{v-k}{k-i}+(-1)^{t+i+1}\sum_{q=0}^{k-t-1}\binom{t-i-1+q}{q}\binom{v-k+q}{k-t}\right\}$$
$$+(\lambda-1)\sum_{j=i}^{t-1}\binom{j}{i}\binom{k}{j}(-1)^{i+j}+\sum_{j=t}^{k-1}\binom{j}{i}w_j(-1)^{i+j},$$

where $x_j \leqslant w_j \leqslant (\lambda-1)\binom{k}{j}$ $(t \leqslant j \leqslant k-1)$.

(The essential part of Lemma 5 is [4, Lemma 6].)

Lemma 6. *For each $k \geqslant 2$ and $l \geqslant 0$, there exist at most finitely many block-regular t-(v, k, λ) designs with $x_i \leqslant l$ for some i $(0 \leqslant i \leqslant t-1)$.*

By making use of Lemmas 5 and 6, we prove Theorem 2.

References

[1] T. Atsumi: An extension of Cameron's result on block-schematic Steiner systems (to appear in J. Combinatorial Theory, ser. A).

[2] R. C. Bose: Strongly regular graphs, partial geometries, and partially balanced designs. Pacific J. Math., **13**, 389–419 (1963).

[3] P. J. Cameron: Two remarks on Steiner systems. Geometriae Dedicata, **4**, 403–418 (1975).

[4] B. H. Gross: Intersection triangles and block intersection numbers for Steiner systems. Math. Z., **139**, 87–104 (1974).