82. *On Certain Densities of Sets of Primes*

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Let $\mathcal{P}$ be the set of all rational primes and M a non-empty subset of $\mathcal{P}$. For a pair of real numbers (α, β), where either $\alpha=0$ and $\beta \geqq 0$ or $\alpha>0$ (β arbitrary), and for positive x, put $f_{\alpha,\beta}(x)=x^{\alpha-1}(\log x)^{\beta}$. We put furthermore

$$\pi_{\alpha,\beta}(M, x)=\sum_{p\in M,\, p\leqq x} f_{\alpha,\beta}(p),$$
$$d_{\alpha,\beta}(M, x)=\frac{\pi_{\alpha,\beta}(M, x)}{\pi_{\alpha,\beta}(\mathcal{P}, x)},$$
$$\underline{D}_{\alpha,\beta}(M)=\liminf_{x\to\infty} d_{\alpha,\beta}(M, x),$$
$$\bar{D}_{\alpha,\beta}(M)=\limsup_{x\to\infty} d_{\alpha,\beta}(M, x).$$

When $\underline{D}_{\alpha,\beta}(M)=\bar{D}_{\alpha,\beta}(M)$, we denote this value by $D_{\alpha,\beta}(M)$ and say that M has the *(α, β)-density* $D_{\alpha,\beta}(M)$. The *natural density* is nothing other than $(1, 0)$-density and, as is well-known, the *Dirichlet density* is equal to $(0, 0)$-density (cf. [1]).

We shall say that (α, β)-density is *stronger* than (γ, δ)-density, and write $D_{\gamma,\delta}\prec D_{\alpha,\beta}$, if the existence of $D_{\alpha,\beta}(M)$ for $M\subset\mathcal{P}$ implies the existence of $D_{\gamma,\delta}(M)$ and, when these densities exist, their values are the same ($\prec$ is obviously an order relation). If $D_{\alpha,\beta}\prec D_{\gamma,\delta}$ and $D_{\gamma,\delta}\prec D_{\alpha,\beta}$, we say that both densities are *equivalent* and write $D_{\alpha,\beta}\sim D_{\gamma,\delta}$ ($\sim$ is clearly an equivalence relation).

Our main theorem states:

Theorem 1. *Any of our (α, β)-densities is equivalent to one of the three densities, $D_{0,0}, D_{0,1}, D_{1,0}$, which will be denoted by d_0, d_1, d_2, respectively. We have furthermore $d_0\prec d_1\prec d_2$ and these three densities are inequivalent.*

As noted above, d_0 and d_2 are Dirichlet density and natural density, respectively. It is known that $d_0\prec d_2$ (cf. [1]). Our theorem shows that the density d_1 lies, so to speak, between the two.

The following theorem gives a more precise form of the first part of Theorem 1.

Theorem 2. *For any $\beta>0$, $D_{0,\beta}$ is equivalent to $d_1=D_{0,1}$ and for any $\alpha>0$ and any β, $D_{\alpha,\beta}$ is equivalent to $d_2=D_{1,0}$.*

Sketch of proof of Theorem 2. It is easily shown that

$$\pi_{\alpha,\beta}(\mathcal{P}, x)=\begin{cases}\left\{\dfrac{1}{\alpha}+o(1)\right\}x^{\alpha}(\log x)^{\beta-1} & \text{if } \alpha>0,\\ \left\{\dfrac{1}{\beta}+o(1)\right\}(\log x)^{\beta} & \text{if } \alpha=0,\ \beta>0.\end{cases}$$

Thus $D_{\alpha,\beta}(M)$ will exist and be equal to μ, if and only if

$$(*)\qquad \pi_{\alpha,\beta}(M, x)=\begin{cases}\left\{\dfrac{\mu}{\alpha}+o(1)\right\}x^{\alpha}(\log x)^{\beta-1} & \text{if } \alpha>0,\\ \left\{\dfrac{\mu}{\beta}+o(1)\right\}(\log x)^{\beta} & \text{if } \alpha=0,\ \beta>0.\end{cases}$$

In the following, we shall limit ourselves to the second part of Theorem 2, as the first part can be proved similarly. Thus we shall suppose $\alpha>0$, $\gamma>0$, and prove $D_{\gamma,\delta}\prec D_{\alpha,\beta}$. Then interchanging the roles of (α, β) and (γ, δ), we obtain $D_{\alpha,\beta}\sim D_{\gamma,\delta}$.

From the assumption that $D_{\alpha,\beta}(M)$ exists, i.e. that the first formula of $(*)$ holds true, we can deduce by partial summation and some computations:

$$\pi_{\gamma,\delta}(M, x)=\left\{\frac{\mu}{\gamma}+o(1)\right\}x^{\gamma}(\log x)^{\delta-1}.$$

$D_{\gamma,\delta}(M)$ exists then and is equal to μ.

Sketch of proof of Theorem 1. Since the relation $d_0\prec d_1\prec d_2$ can be similarly proved to the above, it suffices to show that d_0 and d_1 are inequivalent and so are also d_1 and d_2. This is done by the following two examples.

Example 1. Put

$$M^*=\bigcup_{n=0}^{\infty}\{p\in\mathcal{P};\ \exp((2n)^2)<p\leqq\exp((2n+1)^2)\}.$$

Then we can prove $D_{0,1}(M^*)=1/2$, whereas $\underline{D}_{1,0}(M^*)=0$, $\bar{D}_{1,0}(M^*)=1$.

Example 2. Put

$$M^{**}=\bigcup_{n=0}^{\infty}\{p\in\mathcal{P};\ \exp(\exp(2n))<p\leqq\exp(\exp(2n+1))\}.$$

Then we can prove $D_{0,0}(M^{**})=1/2$, whereas

$$\underline{D}_{0,1}(M^{**})\leqq\frac{1}{e+1},\qquad \bar{D}_{0,1}(M^{**})\geqq\frac{e}{e+1}.$$

Remark. We can show that $D_{1,0}(M)$ (and consequently $D_{\alpha,\beta}(M)$ for any (α, β) treated here) can take any value of $[0,1]$. In fact, the natural density $D_{1,0}(M)$ takes every rational value by Dirichlet's theorem on arithmetic progressions. For irrational μ, take a sequence of positive integers a_ν, b_ν $(\nu=0,1,2,\cdots)$ satisfying $a_\nu>\exp(a_{\nu-1})$, $\varphi(a_\nu)\geqq b_\nu$, and $\lim_{\nu\to\infty} b_\nu/\varphi(a_\nu)=\mu$, where $\varphi(n)$ denotes Euler's function. For each a_ν, take b_ν integers $t_j^{(\nu)}$ $(j=1,2,\cdots,b_\nu)$ which are co-prime to a_ν such that $1\leqq t_j^{(\nu)}<a_\nu$. Put

$$M=\bigcup^{\infty}\{p\in\mathcal{P};\ \exp(a_\nu)<p\leqq\exp(a_{\nu+1}),\ p\equiv t_j^{(\nu)}\ (\mathrm{mod}\ a_\nu)\}.$$

Then it can be shown that $D_{1,0}(M)=\mu$.

Complete proofs are to appear elsewhere.

Reference

[1] H.-H. Ostmann: Additive Zahlentheorie. Bde. I and II, Springer (1956).