## 76. Some Results in the Classification Theory of Compact Complex Manifolds in C

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**0.** By definition a compact complex manifold X is in C if there exist a compact Kähler manifold Z and a surjective meromorphic map  $h: Z \rightarrow X$  [1]. The purpose of this note is then to report some results on the structure of manifolds in C. Details will appear elsewhere.

In what follows X, Y, Z, etc. always denote compact connected complex manifolds in C. We set  $q(X) = \dim H^1(X, O_X)$  and a(X) = the algebraic dimension of X [2]. Let  $f: X \to Y$  be a holomorphic map. For any open subset  $U \subseteq Y$  we write  $X_U = f^{-1}(U)$  and  $f_U = f|_{X_U}$ , and write  $X_v = f^{-1}(y)$  for  $y \in Y$ . We call f a fiber space if f is proper and surjective and has connected fibers. Suppose that f is a fiber space. Then we set dim  $f = \dim X - \dim Y$ , and  $q(f) = q(X_v)$  for any smooth fiber  $X_v$ . Further any fiber space  $f^*: X^* \to Y^*$  which is bimeromorphic to f is called a *bimeromorphic model* of f.

1. Let  $f: X \to Y$  be a fiber space and U a Zariski open subset of Y over which f is smooth. For any integer  $k \ge 0$  we set  $A_k = \{u \in U; a(X_u) \ge k\}$ .

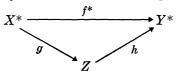
**Proposition 1.**  $A_k$  is a union of at most countably many analytic subsets of U whose closures are analytic in Y.

Let  $a(f) = \max \{k; A_k = U\}$ , so that  $a(X_u) = a(f)$  for 'general'  $u \in U$ . We call a(f) the relative algebraic dimension of f. By Proposition 1 a(f) depends only on the bimeromorphic equivalence class of f. Clearly  $0 \le a(f) \le \dim f$ .

**Proposition 2.** Let  $f: X \rightarrow Y$  and U be as above. Then the following conditions are equivalent. 1)  $a(f) = \dim f$ , 2)  $f_U: X_U \rightarrow U$  is locally Moishezon, and 3) there exists a bimeromorphic model  $f^*: X^* \rightarrow Y^*$  of f which is locally Moishezon.

Here a morphism  $g: X \to Y$  is called *locally Moishezon* if for each  $y \in Y$  there exists a neighborhood  $y \in V$  such that  $g_v: X_v \to V$  is Moishezon, i.e., bimeromorphic over V to a projective morphism.

Definition 1. Let  $f: X \rightarrow Y$  be a fiber space. Then a relative algebraic reduction of f is a commutative diagram



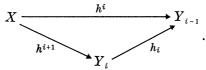
where  $f^*$  is a bimeromorphic model of f and  $a(f) = a(h) = \dim h$ . We also call g a relative algebraic reduction of f. When Y is a point, this reduces to the usual definition of an algebraic reduction of X [2].

**Theorem 1.** Let  $f: X \rightarrow Y$  be any fiber space. Then there exists a relative algebraic reduction of f, and up to bimeromorphic equivalence it is unique.

Applying Theorem 1 successively, for any fiber space f we can find a bimeromorphic model  $f^*: X^* \rightarrow Y^*$  which admits a decomposition;

(1) 
$$X^* \xrightarrow{J^*} Y^* \xrightarrow{Y^*} Y^* \xrightarrow{H_1} Y_0$$

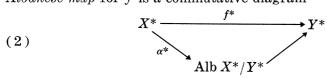
where 1)  $h_i: Y_i \to Y_{i-1}, 1 \leq i \leq m$ , are locally Moishezon, 2) a(g) = 0, and 3) the following commutative diagram is an algebraic reduction of  $h^i = h_i \cdots h_m g$ 



(The case where m=0, or g is isomorphic is included.) Moreover  $f^*$  and the diagram (1) are up to bimeromorphic equivalence uniquely determined by f.

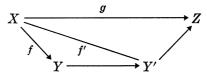
We shall call the diagram (1) the canonical decomposition of f.

2. Definition 2. Let  $f: X \rightarrow Y$  be a fiber space. Then a *relative* Albanese map for f is a commutative diagram



with the following properties; 1)  $f^*$  is a bimeromorphic model of fand 2) there exists a Zariski open subset  $U \subseteq Y^*$  such that  $f^*$  is smooth over U and that for each  $u \in U$  the induced map  $\alpha_u : X_u^* \to \text{Alb } (X^*/Y^*)_u$ is bimeromorphic to the Albanese map  $\alpha(u) : X_u^* \to \text{Alb } X_u^*$ .

Consider a commutative diagram of fiber spaces



where f' is a relative algebraic reduction of g.

**Theorem 2.** In the above situation there exists a relative Albanese map (2) for f for which  $\alpha^*$  is a fiber space. Moreover it is unique up to bimeromorphic equivalence. Corollary. Let  $f: X \rightarrow Y$  be a fiber space. Then there exists a relative Albanese map (2) for f with  $\alpha^*$  a fiber space if either f is an algebraic reduction of X, or a(X)=0.

By Theorem 2  $h_i$  in the canonical decomposition (1) admits a relative Albanese map

(3) 
$$Y_{i}^{*} \xrightarrow{h_{i}^{*}} Y_{i-1}^{*} \qquad 1 \leq i \leq m+1$$

$$(h_{m+1}=g)$$

$$Alb (Y_{i}^{*}/Y_{i-1}^{*}) \qquad (h_{m+1}=g)$$

where  $\alpha_i$  is a fiber space. (For  $h_1$  we have to assume that  $f^*$  is an algebraic reduction of X.) The following two theorems give informations on the structure of  $\mu_1$  and  $\alpha_i$ , while they are of independent interest.

**Theorem 3.** Let  $f: X \rightarrow Y$  be a fiber space. Suppose that f is locally Moishezon and q(f)=0. Then f is Moishezon, i.e., bimero-morphic over Y to a projective morphism.

Let  $f: X \to Y$  be an algebraic reduction of X. Then by Corollary above there exists a relative Albanese map (2) for f with  $\alpha^*$  a fiber space.

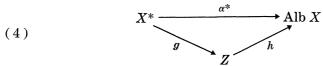
Theorem 4. Suppose that f is locally Moishezon. Then  $q(\alpha^*)=0$ . Corollary. In (3) we have 1) dim  $\mu_i > 0$ , 2)  $q(\alpha_i)=0$ , and 3)  $\alpha_i$  is Moishezon.

We further note that we can take (3) in such a way that  $\alpha_{iu}$  is isomorphic to  $\alpha_i(u)$  in the notation of Definition 2 for  $i \leq m$  (due to the local Moishezonness of  $h_i$ ).

3. Let  $ca(X) = \dim X - a(X)$ . Clearly ca(X) = 0 if and only if X is Moishezon. When ca(X) = 1, the general fiber of the algebraic reduction of X is an elliptic curve, as is well-known.

**Theorem 5.** Suppose that ca(X)=2. Then there exists an algebraic reduction  $f^*: X^* \to Y^*$  of X such that if we denote by  $X^*_y$  any smooth fiber of  $f^*$ , then one of the following holds: 1)  $X^*_y$  is a complex torus, 2)  $X^*_y$  is a K3 surface and  $a(X^*_y)=0$  for general  $y \in Y$ , or 3)  $X^*_y$  is an almost homogeneous ruled surface of genus 1 which is relatively minimal (cf. Remark 12.5 of [2]).

4. We consider manifolds of algebraic dimension zero. So assume that a(X)=0. In this case the Albanese map  $\alpha: X \rightarrow \text{Alb } X$  is a fiber space [2]. In particular  $q(X) \leq \dim X$ . We set  $q^*(X)$  $= \max \{q(\tilde{X}); \tilde{X} \text{ a finite unramified covering of } X\}$  and  $q^{**}(X)$  $= \max \{q(\tilde{X}); \tilde{X} \text{ a finite (ramified) covering of } X\}$ . By the above remark we get  $q(X) \leq q^*(X) \leq q^{**}(X) \leq \dim X$ . We call X primary (specially primary) if  $q(X) = q^*(X) (q(X) = q^{**}(X))$ . Clearly each X has a finite unramified covering X which is primary. Now we consider a relative algebraic reduction (4) of  $\alpha$ :



**Theorem 6.** Suppose that X is primary. Then the following holds true. 1)  $q(\alpha^*)(=q(\alpha))=0$ , 2) there exists a Zariski open subset U of Y such that  $h_U: Z_U \rightarrow U$  is a holomorphic fiber bundle whose typical fiber is a unirational, almost homogeneous projective manifold, and 3) a(g)=0, so that (4) gives the canonical decomposition (1) of  $\alpha$ . Moreover if X is specially primary, then q(g)=0.

Let  $cq(X) = \dim X - q(X)$ . Suppose that X is primary. Then cq(X) > 0. If cq(X) = 1, then  $\alpha$  is a holomorphic  $P^1$  bundle over some Zariski open subset U of Alb X, as follows from the above theorem. In general let  $\kappa(X)$  be the Kodaira dimension of X [2]. Since a(X) = 0,  $\kappa(X) = 0$  or  $-\infty$ .

Theorem 7. Suppose that cq(X)=2. 1) If  $\kappa(X)=0$ , then there exists a finite unramified covering  $\tilde{X}$  of X which is bimeromorphic to a product Alb  $\tilde{X} \times S$  where S is a K3 surface with a(S)=0. 2) If  $\kappa(X) = -\infty$ , then there exists a Zariski open subset  $U \subseteq \text{Alb } X$  such that  $\alpha_U: X_U \rightarrow U$  is a holomorphic fiber bundle whose typical fiber is an almost homogeneous rational surface.

## References

- Fujiki, A.: Closedness of the Douady spaces of compact Kähler spaces. Publ. RIMS, Kyoto Univ., 14, 1-52 (1978).
- [2] Ueno, K.: Classification theory of algebraic varieties and compact complex spaces. Lect. Notes in Math., vol. 439, Springer (1975).

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