71. *Singular Hadamard's Variation of Domains and Eigenvalues of the Laplacian*

By Shin OZAWA
Department of Mathematics, University of Tokyo

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§1. **Introduction.** Let Ω be a bounded domain in $\mathbf{R}^n$ with C^3 boundary γ and w be a fixed point in Ω. For any sufficiently small $\varepsilon>0$, let B_ε be the ball defined by

$$B_\varepsilon=\{z \in \Omega\,;\,|z-w|<\varepsilon\}.$$

Let Ω_ε be the bounded domain defined by $\Omega_\varepsilon=\Omega\backslash\bar{B}_\varepsilon$. Then the boundary of Ω_ε consists of γ and ∂B_ε.

Let $0>\mu_1(\varepsilon)\geq\mu_2(\varepsilon)\geq\cdots$ be the eigenvalues of the Laplacian with the Dirichlet condition on $\gamma\cup\partial B_\varepsilon$. And let $0>\mu_1\geq\mu_2\geq\cdots$ be the eigenvalues of the Laplacian in Ω with the Dirichlet condition on γ. We arrange them repeatedly according to their multiplicities.

The main aim of this note is to give an asymptotic expression of $\mu_j(\varepsilon)$ when ε tends to zero.

We have the following

Theorem 1. *Let Ω be a bounded domain in $\mathbf{R}^2$ with C^3 boundary γ. Fix j. Assume that the multiplicity of μ_j is equal to one, then*

$$\mu_j(\varepsilon)-\mu_j=-2\pi(\log(1/\varepsilon))^{-1}\varphi_j(w)^2+O((\log(1/\varepsilon))^{-2}) \tag{1.1}$$

holds when ε tends to zero. Here φ_j denotes the eigenfunction of the Laplacian with the Dirichlet condition on γ satisfying

$$\int_\Omega \varphi_j(x)^2dx=1.$$

For the case $n=3$, we have the following

Theorem 2. *Let Ω be a bounded domain in $\mathbf{R}^3$ with C^3 boundary γ. Fix j. Assume that the multiplicity of μ_j is equal to one, then*

$$\mu_j(\varepsilon)-\mu_j=-4\pi\varepsilon\varphi_j(w)^2+O(\varepsilon^{3/2}) \tag{1.2}$$

holds when ε tends to zero. Here φ_j denotes the normalized eigenfunction associated with μ_j.

In §2 we give a rough sketch of proof of Theorem 1. To prove Theorem 1 we employ the singular Hadamard variational formula for the Green's function of the Laplacian due to [5]. The details of this paper will be given in [4].

§2. **Outline of proof of Theorem 1.** In this section we give a rough sketch of proof of Theorem 1.

Let $G(x, y)$ be the Green's function on Ω, that is, it satisfies the following:

$$\Delta_x G(x, y) = -\delta(x-y) \qquad x, y \in \Omega$$
$$G(x, y)|_{x \in \gamma} = 0 \qquad y \in \Omega.$$

Fix $y \in \Omega$. Then it is well known that

(2.1) $$\lim_{x \to y} (G(x, y) + (2\pi)^{-1} \log |x-y|) = C_0 < \infty.$$

Fix $w \in \Omega$. For any sufficiently small $\varepsilon > 0$, let ω_ε be the bounded domain defined by

$$\omega_\varepsilon = \{x \in \Omega\,;\, G(x, w) \le (2\pi)^{-1} \log (1/\varepsilon)\}.$$

We put $\beta_\varepsilon = \Omega \backslash \bar{\omega}_\varepsilon$.

Let G_ε and H_ε be a bounded operator in $L^2(\omega_\varepsilon)$ defined by

(2.2) $$(G_\varepsilon f)(x) = \int_{\omega_\varepsilon} G_\varepsilon(x, y) f(y) dy$$

and

(2.3) $$(H_\varepsilon f)(x) = \int_{\omega_\varepsilon} (G(x, y) - 2\pi(\log (1/\varepsilon))^{-1} G(x, w) G(y, w)) f(y) dy$$

for $f \in L^2(\omega_\varepsilon)$ respectively. Here $G_\varepsilon(x, y)$ is the Green's function of the Laplacian in ω_ε. We compare the operators G_ε and H_ε. Put $Q_\varepsilon = H_\varepsilon - G_\varepsilon$. We have the following

Lemma 1. *The equations*

(2.4) $$\begin{aligned} \Delta(Q_\varepsilon f)(x) &= 0 \qquad x \in \omega_\varepsilon \\ (Q_\varepsilon f)(x) &= 0 \qquad x \in \gamma \end{aligned}$$

and

(2.5) $$\max_{x \in \partial \beta_\varepsilon} |Q_\varepsilon f| \le I(\varepsilon) \, \|f\|_{L^2(\omega_\varepsilon)}$$

hold for any $f \in L^2(\omega_\varepsilon)$. Here we put

(2.6) $$I(\varepsilon) = \max_{x \in \partial \beta_\varepsilon} \left(\int_{\omega_\varepsilon} (G(x, y) - G(y, w))^2 dy \right)^{1/2}.$$

We estimate the term $I(\varepsilon)$ as follows:

Lemma 2. *The inequality*

(2.7) $$I(\varepsilon) \le C_1 \varepsilon \, |\log \varepsilon|^{1/2}$$

holds for sufficiently small ε. Here C_1 is a positive constant independent of ε.

In the following $C_2, C_3 \cdots$ are constants independent of ε.

Let $A_{r,\delta}$ be the annulus defined by

$$A_{r,\delta} = \{x \in \mathbf{R}^2\,;\, \delta < |x-w| < r\}.$$

Then it is easy to see that there exists a positive constant q independent of ε such that

(2.8) $$A_{q, \varepsilon/q} \supset \omega_\varepsilon$$

holds.

By Lemma 1, (2.8) and by the maximum principle for harmonic functions we can get the following

Lemma 3. *The inequality*

(2.9) $$\|Q_\varepsilon f\|_{L^2(\omega_\varepsilon)} \le C_2 (\log (1/\varepsilon))^{-1} I(\varepsilon) \, \|f\|_{L^2(\omega_\varepsilon)}$$

holds for any sufficiently small ε.

Let $\tilde{H}_\varepsilon$ be the bounded operator in $L^2(\Omega)$ defined by

$$(2.10)\qquad (\tilde{H}_\varepsilon h)(x)=\int_\Omega (G(x,y)-2\pi(\log(1/\varepsilon))^{-1}G(x,w)G(y,w))h(y)dy$$

for $h\in L^2(\Omega)$.

Now we compare H_ε and $\tilde{H}_\varepsilon$. Let ψ be an eigenfunction of $\tilde{H}_\varepsilon$ satisfying $\|\psi\|_{L^2(\Omega)}=1$ and $\tilde{\lambda}(\varepsilon)$ be its eigenvalue. Then

$$(2.11)\qquad \tilde{H}_\varepsilon\psi=\tilde{\lambda}(\varepsilon)\psi.$$

Let χ_ε be the characteristic function of ω_ε. We put $\psi_1=\chi_\varepsilon\psi$ and $\psi_2=\psi-\psi_1$. For the sake of simplicity, we put

$$h_\varepsilon(x,y)=G(x,y)-2\pi(\log(1/\varepsilon))^{-1}G(x,w)G(y,w).$$

Then (2.11) is equivalent to the following systems of equations (2.12) and (2.13):

$$(2.12)\qquad \int_{\omega_\varepsilon} h_\varepsilon(x,y)\psi_1(y)dy+\int_{\beta_\varepsilon} h_\varepsilon(x,y)\psi_2(y)dy=\tilde{\lambda}(\varepsilon)\psi_1(x)\qquad x\in\omega_\varepsilon$$

$$(2.13)\qquad \int_{\omega_\varepsilon} h_\varepsilon(x,y)\psi_1(y)dy+\int_{\beta_\varepsilon} h_\varepsilon(x,y)\psi_2(y)dy=\tilde{\lambda}(\varepsilon)\psi_2(x)\qquad x\in\beta_\varepsilon.$$

Also we have

$$(2.14)\qquad \|\psi_1\|^2_{L^2(\omega_\varepsilon)}+\|\psi_2\|^2_{L^2(\beta_\varepsilon)}=1.$$

We can get the following

Lemma 4. *The inequality*

$$(2.15)\qquad \left(\int_{\omega_\varepsilon}\left(\int_{\beta_\varepsilon} h_\varepsilon(x,y)\psi_2(y)dy\right)^2dx\right)^{1/2}\le C_3\varepsilon\,\|\psi_2\|_{L^2(\beta_\varepsilon)}$$

holds for any sufficiently small ε.

By Lemma 4 and (2.12), we get

$$(2.16)\qquad \|H_\varepsilon\psi_1-\tilde{\lambda}(\varepsilon)\psi_1\|_{L^2(\omega_\varepsilon)}\le C_3\varepsilon^{1/2}\,\|\psi_2\|_{L^2(\beta_\varepsilon)}.$$

We can also deduce the following inequality (2.17) from (2.13):

$$(2.17)\qquad |\tilde{\lambda}(\varepsilon)|\,\|\psi_2\|_{L^2(\beta_\varepsilon)}\le C_4\varepsilon^{1/2}\,\|\psi_1\|_{L^2(\omega_\varepsilon)}+C_4\varepsilon\,|\log\varepsilon|\,\|\psi_2\|_{L^2(\beta_\varepsilon)}.$$

There exists a positive constant λ^* such that $|\tilde{\lambda}(\varepsilon)|>\lambda^*$ holds for any sufficiently small ε. Therefore by (2.16) and (2.17), we obtain

$$(2.18)\qquad \|(H_\varepsilon-\tilde{\lambda}(\varepsilon))\psi_1\|_{L^2(\omega_\varepsilon)}\le C_5\varepsilon\,\|\psi_1\|_{L^2(\omega_\varepsilon)}$$

and

$$(2.19)\qquad \|\psi_1\|^2_{L^2(\omega_\varepsilon)}\ge 1/2$$

for any sufficiently small ε.

We now study the eigenvalues of G_ε, $\tilde{H}_\varepsilon$ and H_ε. In the first place we compare the eigenvalues of $\tilde{H}_\varepsilon$ and G. It is easily seen that the family $\varepsilon\to\tilde{H}_\varepsilon$ is a holomorphic perturbation family of selfadjoint operators. Therefore we can apply the perturbation theory of eigenvalues in [1] to the pair $\tilde{H}_\varepsilon$ and G. And we have the following

Lemma 5. *Let λ' be a fixed simple eigenvalue of G. Fix small real neighbourhood U of λ'. Then there exists a small positive constant ε_2 such that the following property holds:*

For any $\varepsilon\in(0,\varepsilon_2)$, there exists only one eigenvalue $\lambda'(\varepsilon)$ of $\tilde{H}_\varepsilon$ with multiplicity 1 in U. And $\lambda'(\varepsilon)$ is represented as

$$\lambda'(\varepsilon)=\lambda'-2\pi(\log(1/\varepsilon))^{-1}(\lambda')^2\varphi(w)^2+O((\log(1/\varepsilon))^{-2})$$

when ε tends to 0. *Here $\varphi(x)$ denotes the normalized eigenfunction of G associated with λ'.*

In the next place, we compare the eigenvalues of G_ε and H_ε. Let $0>\tilde{\mu}_1(\varepsilon)\geq\tilde{\mu}_2(\varepsilon)\cdots$ be the eigenvalue of the Laplacian in ω_ε with the Dirichlet condition on $\partial\omega_\varepsilon$. Then by the theorem in [3], we have the following

Lemma 6. *For any fixed j, $\lim_{\varepsilon\to 0}\tilde{\mu}_j(\varepsilon)=\mu_j$. Therefore if μ_j is simple, $\tilde{\mu}_j(\varepsilon)$ is simple for any sufficiently small ε.*

Since there is a correspondence between eigenvalue of the Laplacian and the Green operator, we get the following from Lemma 6.

Lemma 7. *Let λ' be as above. Fix a sufficiently small real neighbourhood V of λ'. Then there exists a constant $\varepsilon_3>0$ depending on V such that the following holds:*

In V, there exists only one eigenvalue $\lambda''(\varepsilon)$ of G_ε for any fixed $\varepsilon\in(0,\varepsilon_3)$.

We see that $\lambda''(\varepsilon)$ is isolated and simple for small ε and $\lim_{\varepsilon\to 0}\lambda''(\varepsilon)=\lambda'$. Therefore by Lemma 3 and a slight modification of theorem in § 134 of [2], we get the following

Lemma 8. *Let λ' be as in Lemma* 5. *And $\lambda''(\varepsilon)$ be as above. Fix a small real neighbourhood V_1 of λ'. Then there exists a constant ε_4 depending on V_1 such that the following hold: For any $\varepsilon\in(0,\varepsilon_4)$, $\lambda''(\varepsilon)\in V_1$. Fix an arbitrary $\varepsilon\in(0,\varepsilon_4)$, then there exists only one eigenvalue $\lambda'''(\varepsilon)$ of H_ε with multiplicity* 1 *in V_1. Moreover,*

$$|\lambda'''(\varepsilon)-\lambda''(\varepsilon)|\leq C_6\varepsilon(\log(1/\varepsilon))^{-1/2}$$

holds.

In the final step, we compare the eigenvalues of $\tilde{H}_\varepsilon$ and H_ε. For this purpose, the following is useful.

Lemma 9. *Let B be a compact selfadjoint operator in a Hilbert space $\mathfrak{H}$. Suppose that the following holds:*

(2.20) *There exists $\eta\in\mathfrak{H}$ such that $\|\eta\|=1$.*

(2.21) *There exists $\lambda^{(4)}\neq 0$, and $\|B\eta-\lambda^{(4)}\eta\|<\varepsilon$ where ε is a sufficiently small positive constant.*

Then there exists at least one eigenvalue $\lambda^{(5)}$ of B in the interval $(\lambda^{(4)}-2\varepsilon,\ \lambda^{(4)}+2\varepsilon)$.

Since we have (2.18) and (2.19), we can apply Lemma 9 to H_ε. Then we get the following

Lemma 10. *Let $\lambda'(\varepsilon)$ be the eigenvalue of $\tilde{H}_\varepsilon$ in Lemma* 5. *Let V_2 be a fixed sufficiently small real neighbourhood of λ'. Then there exists a constant ε_5 depending on V_2 such that the following holds: For any fixed $\varepsilon\in(0,\varepsilon_5)$, there exists at least one eigenvalue of H_ε in the subinterval $(\lambda'(\varepsilon)-C_7\varepsilon,\ \lambda'(\varepsilon)+C_7\varepsilon)$ of V_2.*

We summarize Lemmas 5–10 and we use the relation of eigenvalues of the Laplacian and the Green operators to get the following

Lemma 11. *Fix j. Assume that μ_j is simple, then the relation*

$$\tilde{\mu}_j(\varepsilon)=\mu_j-2\pi(\log(1/\varepsilon))^{-1}\varphi_j(w)^2+O((\log(1/\varepsilon))^{-2})$$

holds when ε tends to zero.

It is easy to see that there exists a positive constant $C>1$ independent of ε such that $\omega_{C\varepsilon}\subset\Omega_\varepsilon\subset\omega_{\varepsilon/C}$ holds for any sufficiently small ε. Since $\tilde{\mu}_j(C\varepsilon)\leq\mu_j(\varepsilon)\leq\tilde{\mu}_j(\varepsilon/C)<0$, and $\tilde{\mu}_j(C\varepsilon)-\tilde{\mu}_j(\varepsilon/C)=O((\log(1/\varepsilon))^{-2})$ when ε tends to zero, then we get Theorem 1.

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