80. P-Convexity with Respect to Differential Operators which act on Linear Subspaces

By Shizuo NAKANE
Department of Mathematics, University of Tokyo

(Communicated by Kôsaku YOSIDA, M. J. A., Nov. 12, 1979)

§1. **Introduction.** We treat linear partial differential operators with constant coefficients of order m:

$$P=P(D),\ D=(D_1,\cdots,D_n),\ D_j=\frac{1}{i}\frac{\partial}{\partial x_j},\ P_m(D)\text{ is its principal part.}$$

We consider the problem of characterizing geometrically the open sets which are P-convex in the case P is independent of some of the variables.

Generally an open set Ω in R^n is called P-convex if for every compact set K in Ω, there exists a compact set K' in Ω such that for every u in $\mathcal{E}'(\Omega)$

$$\operatorname{supp} P(-D)u\subset K \quad \text{implies} \quad \operatorname{supp} u\subset K'.$$

The importance of this concept lies in the following property of P-convex sets proved by B. Malgrange [3]: The equation $P(D)u=f$ in Ω has a C^∞ solution u for every f in $C^\infty(\Omega)$ if and only if Ω is P-convex.

It is well known that an open convex set is P-convex for every differential operator P. However, complete characterizations of P-convexity are known only in the following cases:

1) P is elliptic (L. Hörmander [1]),
2) $n=2$ (L. Hörmander [1]),
3) P is of first order (E. C. Zachmanoglou [10]),
4) $n=3$, P is of principal type, i.e. $P_m(\xi)=0$ implies grad $P_m(\xi)\neq 0$ and $\partial\Omega$ is C^2 (J. Persson [8]),
5) $P(D)=D_1D_2+\frac{1}{2}\sum_{j=3}^{n} D_j$, and $\partial\Omega$ is C^2 (J. Persson [9]).

We restrict our attention to the operators as above, and we consider them as operators in R^{n+k}. Consequently, they are independent of the variables $(x_{n+1},\cdots,x_{n+k})$.

Under these somewhat restricted situations, we obtain a sufficient condition for P-convexity with respect to such operators. Especially, in the case 1), we show that the sufficient condition is also necessary.

In §2, we shall give the definition and the properties of uniqueness cones, and in Theorem 1 we shall obtain a sufficient condition for P-convexity in general cases. In §3, we shall treat the operators 1)–5) in R^{n+k}, and obtain a sufficient condition for P-convexity in such cases. And we shall show that in the case 4) the hypothesis that P is of principal type can be relaxed to constant multiplicity.

§ 2. **Results in general cases.** The trick we use lies in the uniqueness cones which are introduced in J. Persson [5].

Definition 1. Let M be a proper open convex cone in R^n with vertex 0. For an element y of R^n, we set

$$K(M, y)=\{x \in R^n : \langle x-y, \xi\rangle \leqslant 0 \text{ for every } \xi \in M\}.$$

And let N be an element of M. Then we set

$$K(N, M, y, \rho)=\{x \in K(M, y) : \langle x-y, N\rangle \geqslant -\rho\}, \qquad 0<\rho\leqslant\infty.$$

If M is contained in $\{\xi \in R^n : P_m(\xi)\neq 0\}$, we call $K(N, M, y, \rho)$ *a uniqueness cone of P at y.*

Then we have the following lemma obtained in J. Persson [5].

Lemma 1. *Let u be an element of $\mathcal{D}'(\Omega)$, and let $K(N, M, y, \rho)$ be a uniqueness cone of P which is contained in Ω. Let*

$$P(D)u=0 \qquad \textit{near } K(N, M, y, \rho),$$

$$(1) \quad u=0 \textit{ near } K(N, M, y, \rho)\cap\{x : \langle x-y, N\rangle=-\rho\}.$$

Then,

$$u=0 \qquad \textit{in } \mathrm{Int}\, K(N, M, y, \rho).$$

(*Here we call the set in* (1) *a bottom of the cone.*)

The proof follows from Holmgren's uniqueness theorem. This lemma shows that the zeros of the distribution solutions of the equation $P(D)u=0$ propagate along the uniqueness cones of P. Especially, if P is elliptic, M can be chosen arbitrarily close to an open half space. So uniqueness cones of P can be chosen arbitrarily close to segments or half lines, and the zeros of the solutions propagate along every segment. We remark that this lemma is true even if u is a hyperfunction, since Holmgren's theorem is true for hyperfunctions. So all the results on unique continuation of solutions as follows are true for hyperfunctions.

Next, by a chain of cones, we mean a finite set of cones $\{K_i\}$, such that the bottom of K_i is contained in K_{i-1}.

Definition 2. Let K be a compact subset of Ω. We set $\hat{K}(P, \Omega)=K\cup\{x \in K^c : x$ cannot be connected to Ω^c or to the infinity by a chain of uniqueness cones of P in K^c.$\}$, and we call it *the weak P-hull of K in Ω.*

Especially, if P is elliptic, $\hat{K}(P, \Omega)$ is the compact subset of Ω consisting of the union of K and all connected components of K^c which are relatively compact in Ω. The following lemma is a direct consequence of Lemma 1.

Lemma 2. *Let K be a compact subset of Ω, and let u be an element in $\mathcal{E}'(\Omega)$. Then*

$$\mathrm{supp}\, P(-D)u\subset K \quad \textit{implies} \quad \mathrm{supp}\, u\subset\hat{K}(P, \Omega).$$

Now we can give a sufficient condition for P-convexity.

Theorem 1. *If $\hat{K}(P, \Omega)$ is compact in Ω for every compact subset K of Ω, then Ω is P-convex.*

We emphasize that in the cases 1)–5), this sufficient condition given in Theorem 1 is also necessary, i.e. in these cases,

(2) *Ω is P-convex if and only if $\hat{K}(P, \Omega)$ is compact in Ω for every compact subset K of Ω.*

§ 3. Results in the case P is independent of some of the variables. From now on, we consider the operator $P(D)$ in R^{n+k} such that P is independent of $(x_{n+1}, \cdots, x_{n+k})$, i.e. P acts on a linear subspace R^n of R^{n+k}. To avoid confusion, when we consider P as an operator in R^n, we denote it by P'. And we investigate P-convexity of an open set Ω in R^{n+k}. We use the following notations:

$$R^{n+k} \ni x=(x', x'') \in R^n \times R^k,$$

$$\Omega_a=\{x \in \Omega : x''=a\},\ K_a=\{x \in K : x''=a\}, \qquad a \in R^k.$$

Here K is a compact subset of Ω.

Now we obtain a necessary condition for P-convexity. In the following statements, when we consider P'-convexity of Ω_a, we look upon Ω_a as an open set in R^n.

Theorem 2. *Let $P'(D')$ be a differential operator in R^n, and we denote it by $P(D)$ when we consider it in R^{n+k}. Then, if Ω is P-convex, Ω_a is P'-convex for every a in R^k.*

Proof. We show that if for an a in R^k Ω_a is not P'-convex, Ω is not P-convex.

If Ω_a is not P'-convex, there exist a compact subset K_a of Ω_a and a sequence $u_j(x')$ in $\varepsilon'(\Omega_a)$ such that

$$\operatorname{supp} P'(-D')u_j \subset K_a \quad \text{and} \quad \operatorname{dist}(\operatorname{supp} u_j, \Omega_a^c) \to 0 \qquad \text{if } j \to \infty.$$

So we set

$$v_j(x)=u_j(x')\delta(x''-a).$$

Then v_j is an element of $\varepsilon'(\Omega)$ such that

$$\operatorname{supp} P(-D)v_j \subset K_a \quad \text{and} \quad \operatorname{dist}(\operatorname{supp} v_j, \Omega^c) \to 0 \qquad \text{if } j \to \infty.$$

Consequntly Ω is not P-convex.

From the case P' is elliptic in R^n, we can easily show that this condition is not sufficient. But we obtain a sufficient condition for P-convexity if P' satisfies (2) in R^n.

Theorem 3. *Let P' satisfy (2) in R^n, and let Ω be an open set in R^{n+k}. We assume for every a in R^k Ω_a is P'-convex and $\bigcup_a \hat{K}_a(P', \Omega_a)$ is compact in Ω. Then Ω is P-convex.*

Proof. It easily follows from definition that a uniqueness cone of P at y can be chosen arbitrarily close to an n-dimensional cone in $R^{n+k} \cap \{x : x''=y''\}$ which is identical with the uniqueness cone of P' at y' in $R^{n+k} \cap \{x : x''=y''\}$, i.e. which is parallel to R^n. Consequently, $\hat{K}(P, \Omega)$ is identical with $\bigcup_a \hat{K}_a(P', \Omega_a)$ and is compact in Ω from the hypothesis. So by Theorem 1, Ω is P-convex.

Especially if P' is elliptic in R^n, the hypothesis in Theorem 3 is also necessary. Consequently we obtain a complete characterization of P-convexity in this case.

Theorem 4. *Let P' be elliptic in R^n, and let Ω be an open set in R^{n+k}. Then Ω is P-convex if and only if Ω satisfies the hypothesis of Theorem* 3.

The proof is a slight modification of that of Theorem 4 in E. C. Zachmanoglou [10]. In [10], he treated the case P is of first order, but the argument in [10] is applicable to this case. See [10].

Proof. We have only to prove the necessity. Suppose that for a compact subset K of Ω, $\bigcup_a \hat{K}_a(P', \Omega_a)$ is not compact. Then there is a sequence $\{x^i\}$ of points in $\bigcup_a \hat{K}_a(P', \Omega_a)$ converging to a point x^0 belonging to $[\bigcup_a \hat{K}_a(P', \Omega_a)]^c$. Since K is compact, we may assume that $\{x^i\}$ is contained in K^c. Let W_i be the connected component of $K^c \cap \{x \in R^{n+k} : x'' = x^{i\prime\prime}\}$ which is relatively compact in Ω and which contains x^i. Let W_0 be the connected component of $K^c \cap \{x \in R^{n+k} : x'' = x^{0\prime\prime}\}$ which contains x^0. Since x^0 belongs to $[\bigcup_a \hat{K}_a(P', \Omega_a)]^c$, W_0 must be either unbounded or it must intersect Ω^c. If W_0 is unbounded, x^0 is the end point of an unbounded polygonal path which lies in W_0. This path is closed and does not intersect K, hence its distance from K is positive. This clearly implies that for sufficiently large i, W_i is also unbounded. This is a contradiction, hence W_0 must intersect Ω^c. So it easily follows that dist (W_i, Ω^c) tends to zero. We remark that ∂W_i is contained in K.

We shall find u_i in $\mathcal{E}'(\Omega)$ such that

$$\operatorname{supp} u_i = \overline{W}_i \quad \text{and} \quad \operatorname{supp} P(-D)u_i \subset W_i.$$

Then Ω is not P-convex and the proof is completed. We set

$$v_i(x'') = \delta(x'' - x^{i\prime\prime}).$$

Then

$$\operatorname{supp} v_i = \{x \in R^{n+k} : x'' = x^{i\prime\prime}\} \quad \text{and} \quad P(-D)v_i = P'(-D')v_i = 0.$$

Let $w(x')$ be an analytic solution of $P'(-D')w(x') = 0$, and we set

$$u_i(x) = \chi_i(x')w(x')v_i(x''),$$

where χ_i is the characteristic function of W_i in R^n. Then u_i satisfies the properties we have mentioned above.

Obviously from what we have mentioned, Theorem 3 can be applied also to the cases 2)–5). But the auther doesn't know whether the hypothesis of Theorem 3 is necessary or not in these cases.

Finally we remark that in the case 4), the hypothesis P is of principal type can be relaxed to constant multiplicity. In fact, uniqueness cones are defined independently of mutiplicity and null solutions can be constructed in constant multiplicity case. See J. Persson [6], [7] or H. Komatsu [2].

References

[1] L. Hörmander: Linear Partial Differential Operators. Springer, Berlin (1963).

[2] H. Komatsu: Irregularity of characteristic elements and construction of null solutions. J. Fac. Sci. Univ. Tokyo Sec. IA, **23**, 297–342 (1976).

[3] B. Malgrange: Existence et approximation des solutions des équations aux dérivées et des équations de convolutions. Ann. Inst. Fourier, **6**, 271–355 (1955–1956).

[4] J. Persson: A sufficient condition for a set to be P-convex in R^n. Mathematiche, **25**, 1–7 (1970).

[5] ——: On uniqueness cones, velocity cones and P-convexity. Ann. Mat. Pura Appl., **96**(4), 69–87 (1973).

[6] ——: Semi-global null solutions and P-convexity. Boll. Un. Mat. Ital., **8**(4), 20–28 (1973).

[7] ——: Correction of "Semi-global null solutions and P-convexity". Ibid., **11**(4), 518–524 (1975).

[8] ——: The Cauchy problem at simply characteristic points and P-convexity (to appear in Ann. Mat. Pura Appl.).

[9] ——: The wave operator and P-convexity (to appear).

[10] E. C. Zachmanoglou: An application of Holmgren's uniqueness theorem and convexity with respect to differential operators with flat characteristic cones. Trans. Amer. Math. Soc., **140**, 109–115 (1969).